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$1/J^2$ corrections to BMN energies from the quantum long range Landau-Lifshitz model

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Abstract: In a previous paper [17], it was shown that quantum $1/J$ corrections to the BMN spectrum in an effective Landau-Lifshitz (LL) model match with the results from the one-loop gauge theory, provided one chooses an appropriate regularization. In this paper we continue this study for the conjectured Bethe ansatz for the long range spin chain representing perturbative large-N $\mathcal{N}=4$ Super Yang-Mills in the SU(2) sector, and the "quantum string" Bethe ansatz for its string dual. The comparison is carried out for corrections to BMN energies up to order $\tilde{\lambda}^3$ in the effective expansion parameter $\tilde{\lambda}=\lambda/J^2$. After determining the "gauge-theory" LL action to order $\tilde{\lambda}^3$, which is accomplished indirectly by fixing the coefficients in the LL action so that the energies of circular strings match with the energies found using the Bethe ansatz, we find perfect agreement. We interpret this as further support for an underlying integrability of the system. We then consider the "string-theory" LL action which is a limit of the classical string action representing fast string motion on an S^3 subspace of S^5 and compare the resulting $\tilde{\lambda}^3/J^2$ corrections to the prediction of the "string" Bethe ansatz. As in the gauge case, we find precise matching. This indicates that the LL hamiltonian supplemented with a normal ordering prescription and ζ-function regularization reproduces the full superstring result for the $1/J^2$ corrections, and also signifies that the string Bethe ansatz does describe the quantum BMN string spectrum to order $1/J^2$. We also comment on using the quantum LL approach to determine the non-analytic contributions in λ that are behind the strong to weak coupling interpolation between the string and gauge results.

Keywords: Bethe Ansatz, Integrable Field Theories, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

Quantum corrections to semiclassical solutions of strings propagating on $AdS_5 \times S^5$ play an important part in the investigation of AdS/CFT duality [1−3]. In particular, the so-called three loop discrepancy between gauge and string predictions was first found when computing the leading $1/J$ correction[1] to the two-impurity BMN state [5]. The discrepancy was later found [6] to be present also for the semiclassical spinning string solutions [7].

[1]For string computations, $1/\sqrt{\lambda}$ acts as an inverse string tension or as $\hbar$, which for semiclassical strings with large total R-charge J, can formally be traded with $1/J$. However, it turns out that at higher orders of perturbation theory there are additional genuine quantum corrections [4] which are reflected in the presence of terms non-analytic in λ/J^2.

The conclusion of [5] (see also [8]) was the result of a complicated calculation, and used the contributions from the full set of world sheet fields, both bosonic and fermionic. Likewise, the quantum superstring $1/J$ corrections were computed for the circular string solution of [7], again employing the full set of the bosonic and fermionic world-sheet fields [9–11].

In the gauge theory, once one has found the dilatation operator, the fermionic excitations are not needed to compute $1/J$ corrections in the SU(2) or the SU(1, 1) sectors, which are closed sectors containing no fermion fields.[2] At the one-loop level, that is to linear order in $\tilde{\lambda}$, where $\tilde{\lambda}$ is the effective coupling $\tilde{\lambda} = \lambda/J^2$, the corrections can be determined from the corresponding Bethe ansätze for these sectors [12, 13]. At higher loops, one can use the proposed long range Bethe ansätze in [6, 14, 15].

Since fermions seem to play no role in the Bethe ansatz, we should also be able to ignore them when computing $1/J$ corrections from an effective action. In [16] and our previous paper [17] this was shown to be the case when computing $1/J$ corrections for the one-loop SU(2) sector. The action used was the Landau-Lifshitz (LL) action, which is the effective action for the ferromagnetic Heisenberg spin chain in the continuum limit, with higher derivative counterterms to account for lattice effects. On the string side, the counterpart of this action is the fast string limit around an S^3 subspace of S^5 [18–20]. However, with only bosonic SU(2) sector modes being quantized, i.e. without the rest of the superstring modes, including fermions, there are infinities that need to be regularized. This can be accomplished with a combination of normal ordering and ζ-function regularization.

A natural extension of [17] is to carry out the computations for higher orders in $\tilde{\lambda}$. In terms of the spin chain, this corresponds to going beyond nearest neighbor interactions, with order $\tilde{\lambda}^n$ contributions coming from interactions between spins separated by up to n sites. In [6], Serban and Staudacher (SS) first proposed an all-loop Bethe ansatz that was based on the Inozemtsev spin chain [21] and correctly reproduced the two and three loop predictions in [22]. However, this Bethe ansatz violated the BMN scaling at the 4 loop level, so a different ansatz was proposed by Beisert, Dippel and Staudacher (BDS) [14] that produces identical results as the SS ansatz to order $\tilde{\lambda}^3$, but also preserves BMN scaling to all loops in the thermodynamic limit.

In order to compare results between a long range Bethe ansatz calculation and an effective action calculation, we need to find the relevant extension of the LL action. The effective LL action to $\tilde{\lambda}^2$ order was derived, both from the spin chain hamiltonian and the fast string limit, in [19]. To go beyond $\tilde{\lambda}^2$ order on the gauge side, the "string" LL action that follows from the fast string limit [19] can no longer be used, since the results on the gauge and string sides are known to disagree.

In this paper, we are able to find the "gauge" LL action to $\tilde{\lambda}^3$ order indirectly, by using the results from the SS/BDS Bethe ansatz for operators that are dual to circular strings [23]. We construct the LL action by including all possible six derivative terms and varying their coefficients so that the energies agree with the Bethe ansatz predictions.

[2] In deriving the expression for the dilatation operator one of course uses the full set of bosonic and fermionic fields of the SYM theory (for example, already at one loop, fermions contribute to the scalar self-energy diagrams).

With an effective action now available, we can then directly compute the $1/J^2$ corrections to BMN states with M impurities up to $\tilde{\lambda}^3$ order by quantizing the LL action (assuming normal ordering of the hamiltonian and using a ζ-function regularization to remove further infinities). Remarkably, comparing the results to the ones found directly from the gauge theory Bethe ansatz, we find perfect agreement.

One can also do the same on the string side, although in some sense the logic is in the reverse direction. Here one starts with the string effective action on $R \times S^3$ and takes the fast string limit, reducing the action to a "string" LL effective action. Results from the 'string LL action can then be compared with results from the string "quantum" Bethe ansatz of Arutyunov, Frolov and Staudacher (AFS) [24], which itself was originally derived by "discretising" the equations in [25] for general classical string motion on $R \times S^3$. Again, for the $1/J^2$ corrections for M-impurity BMN states we find, even more remarkably, perfect agreement up to $\tilde{\lambda}^3$ order.

That these results match attests to the underlying integrability of these systems. Any system, integrable or otherwise, should be describable by an effective action. However, the presence of a Bethe ansatz means that all scattering amplitudes can be reduced to products of two body scattering, the hallmark of integrability. Our results seem to indicate that the effective actions we use are consistent with integrability, and that this integrability will be present at the quantum level (or at least the first two orders), even for the string theory.

On the gauge side, the LL action should be interpreted strictly through its series expansion in $\tilde{\lambda}$, since the 't Hooft coupling λ is the natural perturbative expansion parameter. However, for the string theory, $1/\sqrt{\lambda}$ is the natural semiclassical parameter and so for the string LL action we are formally allowed to expand in $1/J$ while keeping $\tilde{\lambda}$ fixed. Hence, when determining the $1/J$ corrections, on the gauge side, one should first expand in $\tilde{\lambda}$ and then compute the quantum corrections, while, on the string side, one should first compute the quantum corrections and then expand in $\tilde{\lambda}$. Because of the divergences that arise, the two procedures do not commute and may lead to different results. In particular, for the string theory, this will lead to non-analytic terms in $\tilde{\lambda}$ [4, 26] and such non-analytic terms should be included only within the "string" interpretation of the LL computation.

We should stress that the presence of such non-analytic terms in the near-BMN spectrum is non-trivial: on general grounds one expects the energy to have the following expansion

$$E = h_0(\tilde{\lambda}) + \frac{h_1(\tilde{\lambda})}{J} + \frac{h_2(\tilde{\lambda})}{J^2} + \cdots\ , \tag{1.1}$$

and while h_0 and h_1 are known to have a regular expansion in integer powers of $\tilde{\lambda}$, the results of [4] suggest that h_2 should contain non-analytic terms with half-integer powers of $\tilde{\lambda}$ starting with $\tilde{\lambda}^{5/2}$. Below we will look for such non-analytic terms in the BMN spectrum using the quantum string LL approach, with mixed results. Indeed, we do find half integer powers of $\tilde{\lambda}$ in the $1/J^2$ corrections computing from string LL hamiltonian, but these come with logarithmic divergences that needed to be regularized. Presumably, for the full superstring calculation the coefficients of these non-analytic terms will be finite, but we are unable to unambiguously find these finite contributions using the string LL action.

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This paper is organized as follows: section 2 describes the structure of the string and gauge LL actions, with the latter determined to $\tilde{\lambda}^3$ order by comparing to results from the Bethe ansatz. Section 3 is a review of the quantization procedure developed in [17] relevant for BMN calculations, now applied to an LL action of more general structure. Sections 4 and 5 contain computations respectively for the $1/J$ and $1/J^2$ corrections. Section 6 discusses non-analytic corrections and section 7 contains some concluding remarks. Appendix A describes how to fix the structure of the gauge LL action to $\tilde{\lambda}^3$ order. Appendix B presents computations of the energy of M impurity BMN states to $1/J^2$ order from both the gauge and string Bethe ansätze. Appendix C discusses the structure of non-analytic terms in the 1-loop energy of a circular string solution, suggesting the interpretation of the observations in section 6.

2. Classical LL action to $\tilde{\lambda}^3$ order

Let us start by describing the structure of the LL action viewed as an effective action for low-energy excitations on either the string or gauge theory side (for a review see [27, 20, 17]). On the gauge side it is understood in a perturbative expansion in $\tilde{\lambda} = \frac{\lambda}{J^2}$, and represents the quantum effective action for the low-energy spin wave modes of the spin chain hamiltonian equivalent to the perturbative planar dilatation operator in the SU(2) sector [12]. On the string side it is a "fast-string" expansion of the classical string action in a gauge [19, 20] where the density of the momentum of the "fast" collective coordinate is constant.

It is known [19] that to "2-loop" or $\tilde{\lambda}^2$ order the LL actions obtained from the string theory and gauge theory are the same. At "3-loop" or $\tilde{\lambda}^3$ order, however, they are different. The coefficients in the string LL action were obtained in [19] while for the gauge-theory LL action one can fix them by comparing the energy of particular classical solutions with the one obtained from the spin chain Bethe ansatz. We shall discuss this in appendix A.

As a result, one may write the LL action in the SU(2) sector as (we use the gauge where $t = \tau;\ \partial_1 \equiv \partial_\sigma$)

$$S = J \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} L \,, \tag{2.1}$$

where the lagrangian is

$$\begin{aligned} L = \vec{C}(n) \cdot \partial_t \vec{n} - \frac{1}{4} \vec{n} \left(\sqrt{1 - \tilde{\lambda} \partial_1^2} - 1 \right) \vec{n} - \frac{3\tilde{\lambda}^2}{128} (\partial_1 \vec{n})^4 - \\ - \frac{\tilde{\lambda}^3}{64} \left[\mathbf{a} (\partial_1 \vec{n})^2 (\partial_1^2 \vec{n})^2 + \mathbf{b} (\partial_1 \vec{n} \partial_1^2 \vec{n})^2 + \mathbf{c} (\partial_1 \vec{n})^6 \right] + O(\tilde{\lambda}^4) \,. \end{aligned} \tag{2.2}$$

Here $dC = \epsilon^{ijk} n_i dn_j \wedge dn_k$, i.e. $\vec{C}$ is a monopole potential on S^2. Also $\tilde{\lambda} = \lambda/J^2$, where $\lambda \sim 1/(\alpha')^2$. Here $\vec{n}(t,\sigma)$ is a unit vector, and we have included all terms which are quadratic in $\vec{n}$. This exact quadratic part follows from the string action [19] and also from the coherent-state expectation value of the spin-spin part of the dilatation operator on the gauge theory side [28]. It reproduces the BMN dispersion relation for small ("magnon") fluctuations near the BPS vacuum $\vec{n} = (0, 0, 1)$.

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The values of the "3-loop" coefficients in the string and gauge theory expressions for (2.2) are:

$$\mathbf{a}_s = -\frac{7}{4}\,, \qquad \mathbf{b}_s = -\frac{25}{2}\,, \qquad \mathbf{c}_s = \frac{13}{16}\,, \tag{2.3}$$

$$\mathbf{a}_g = -\frac{7}{4}\,, \qquad \mathbf{b}_g = -\frac{23}{2}\,, \qquad \mathbf{c}_g = \frac{3}{4}\,. \tag{2.4}$$

The string coefficients were found in [19]. The gauge coefficients $\mathbf{a}$ and $\mathbf{c}$ are fixed by comparing to the Bethe ansatz results for the circular solution (see appendix A), while the coefficients $\mathbf{a}$ and $\mathbf{b}$ can be fixed by matching the resulting $\tilde{\lambda}^3/J$ correction to the BMN energy to the corresponding gauge Bethe ansatz result [14, 24] (see section 4).[3] We shall see in section 5 that with these coefficients the $\tilde{\lambda}^3/J^2$ corrections also match, which provides a strong consistency check.

The difference between the string and gauge values of the coefficients $\mathbf{b}$ and $\mathbf{c}$ implies the difference between the LL lagrangians or the hamiltonians

$$L_s - L_g = -(H_s - H_g) = \frac{\tilde{\lambda}^3}{64}\left[(\partial_1\vec{n}\cdot\partial_1^2\vec{n})^2 - \frac{1}{16}(\partial_1\vec{n}\cdot\partial_1\vec{n})^3\right] + O(\tilde{\lambda}^4)\,. \tag{2.5}$$

This is a manifestation of the "3-loop disagreement" [5, 6]. Following [4], it can explained by promoting the coefficients $\mathbf{b}$ and $\mathbf{c}$ to functions of λ such that for large λ they approach the string theory values, while for small λ they approach the gauge theory values. Subleading terms in the string (strong-coupling) expansion of $\mathbf{b}(\lambda)$ and $\mathbf{c}(\lambda)$,

$$\mathbf{b}(\lambda) = \mathbf{b}_s + \frac{b_1}{\sqrt{\lambda}} + \cdots\,, \qquad \mathbf{c}(\lambda) = \mathbf{c}_s + \frac{c_1}{\sqrt{\lambda}} + \cdots\,, \qquad \lambda \gg 1\,, \tag{2.6}$$

should come from the part of the string quantum corrections which are non-analytic in $\tilde{\lambda}$ [4]. We shall return to the discussion of this below.

As in [17], let us now rewrite the LL lagrangian (2.2) in terms of two independent fields. Solving the constraint $|\vec{n}|^2 = 1$ as $n_3 = \sqrt{1 - n_1^2 - n_2^2}$ we get the following SO(2) invariant expression for the lagrangian in terms of n_1 and n_2 ($a, b = 1, 2;\ \ n^2 = n_a n_a$)

$$L = h^2(n)\epsilon_{ab}\dot{n}_a n_b - H(n_1, n_2)\,, \qquad h^2(n) = \frac{1 - \sqrt{1 - n^2}}{2n^2} = \frac{1}{4} + \frac{1}{16}n^2 + \cdots \tag{2.7}$$

$$\begin{aligned}
H(n_1, n_2) = {} & \frac{1}{4}n_a\left(\sqrt{1 - \tilde{\lambda}\partial_1^2} - 1\right)n_a + \frac{1}{4}\sqrt{1 - n^2}\left(\sqrt{1 - \tilde{\lambda}\partial_1^2} - 1\right) \times \\
& \times \sqrt{1 - n^2} + \frac{3\tilde{\lambda}^2}{128}\left[n_a'^2 + \frac{(n_a n_a')^2}{1 - n^2}\right]^2 + \\
& + \frac{\tilde{\lambda}^3}{64}\left\{\mathbf{a}\left[n_a'^2 + \frac{(n_a n_a')^2}{1 - n^2}\right] \times \right. \\
& \qquad \times \left[n_a''^2 + \frac{\left(n_a n_a''(1 - n^2) + n'^2 - n_a^2 n_b'^2 + (n_a n_a')^2\right)^2}{(1 - n^2)^3}\right] +
\end{aligned}$$

[3] In principle, to fix the values of the three coefficients we could use, instead of the quantum $1/J$ BMN corrections, the classical folded string solution, comparing its LL energy to the Bethe ansatz result of [6].

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$$+\mathbf{b}\Big[n'_a n''_a + \frac{n_a n'_a}{(1-n^2)^2}\Big(n_a n''_a(1-n^2) + n'^2 - n_a^2 n_b'^2 + (n_a n'_a)^2\Big)\Big]^2 +$$

$$+\mathbf{c}\Big[n_a'^2 + \frac{(n_a n'_a)^2}{1-n^2}\Big]^3\Big\} + O(\tilde\lambda^4)\,, \tag{2.8}$$

where we use dot and prime for world-sheet time and space derivatives. The function $h(n)$ has a regular expansion near $n_a = 0$, and so (2.8) may be interpreted as a phase-space lagrangian with, say, n_1 being a coordinate and n_2 being related to its momentum.

To simplify the quantization of the LL lagrangian near a particular solution it is useful to put it into the standard canonical form [17] by doing the field redefinition $n_a \to z_a$

$$z_a = h(n)\; n_a\,, \qquad n_a = 2\sqrt{1-z^2}z_a\,, \tag{2.9}$$

to obtain

$$L = \epsilon_{ab}\dot z_a z_b - H(z_1, z_2)\,. \tag{2.10}$$

Having the lagrangian in the standard form $L = p\dot q - H(p,q)$, the quantization is straightforward: we promote z_a to operators, impose the canonical commutation relation (cf. (2.1))

$$[z_1(t,\sigma), z_2(t,\sigma')] = iJ^{-1}\pi\delta(\sigma-\sigma')\,, \tag{2.11}$$

and then decide how to order the "coordinate" and "momentum" operators in $H(z_1, z_2)$.

3. Quantization near BPS vacuum: corrections to BMN spectrum from LL hamiltonian

As in [17] our aim will be to use the LL action to compute quantum $1/J$ and $1/J^2$ corrections to the BMN spectrum of fluctuations near the BPS vacuum solution

$$\vec n = (0,0,1)\,, \tag{3.1}$$

representing the massless geodesic in $R_t \times S^3$. The $1/J$ corrections can be found from the Bethe ansatz on the spin chain [12, 29] or from a direct superstring quantization [8, 5]. As explained in [17], the derivation from the LL action turns out to be much simpler than the string-theory derivation. Here we shall extend the method of [17] to $\tilde\lambda^3/J$ and $\tilde\lambda^3/J^2$ orders.[4] The $1/J^2$ corrections to the BMN spectrum have not yet been obtained from a full superstring computation, and our LL approach provides a useful short-cut, highlighting several important issues that will also appear in the exact superstring approach.

Expanding near this vacuum corresponds to expansion near $n_a = 0$ in (2.8) or $z_a = 0$ in (2.10). Observing that the factor J in front of the LL action (2.1) plays the role of the inverse Planck constant, it is natural to rescale z_a as

$$z_1 = \frac{1}{\sqrt{J}}f\,, \qquad z_2 = \frac{1}{\sqrt{J}}g\,, \tag{3.2}$$

[4]Unfortunately, the exact (all order in $\tilde\lambda$) form of the $\vec n^4$ and $\vec n^6$ terms in the LL action is not known, preventing us from computing the $1/J$ and $1/J^2$ corrections to all orders in $\tilde\lambda$.

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so that powers of $1/J$ will play the role of coupling constants for the fluctuations in the non-linear LL hamiltonian. Expanding the hamiltonian in (2.8), (2.10) to sixth order in the fluctuation fields f, g we get

$$S = \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} \, (2g\dot{f} - H) \,, \qquad H = H_2 + H_4 + H_6 + \cdots \,, \tag{3.3}$$

$$H_2 = f\Big(\sqrt{1 - \tilde{\lambda}\partial_1^2} - 1\Big) f + g\Big(\sqrt{1 - \tilde{\lambda}\partial_1^2} - 1\Big) g \,, \tag{3.4}$$

$$\begin{aligned} H_4 = \frac{1}{J}\bigg\{ & (f^2+g^2)\Big(\sqrt{1 - \tilde{\lambda}\partial_1^2} - 1\Big)(f^2+g^2) - f(f^2+g^2)\Big(\sqrt{1 - \tilde{\lambda}\partial_1^2} - 1\Big) f - \\ & - g(f^2+g^2)\Big(\sqrt{1 - \tilde{\lambda}\partial_1^2} - 1\Big) g + \frac{3\tilde{\lambda}^2}{8}(f'^2+g'^2)^2 + \\ & + \frac{\tilde{\lambda}^3}{4}\left[\mathbf{b}(f'f''+g'g'')^2 + \mathbf{a}(f'^2+g'^2)(f''^2+g''^2)\right] \bigg\} + O\Big(\frac{\tilde{\lambda}^4}{J}\Big) \,, \end{aligned} \tag{3.5}$$

$$\begin{aligned} H_6 = \frac{1}{J^2}\bigg\{ & \frac{1}{4} f(f^2+g^2)\Big(\sqrt{1 - \tilde{\lambda}\partial_1^2} - 1\Big)[f(f^2+g^2)] + \\ & + \frac{1}{4} g(f^2+g^2)\Big(\sqrt{1 - \tilde{\lambda}\partial_1^2} - 1\Big)[g(f^2+g^2)] - \\ & - \frac{1}{4} f(f^2+g^2)^2\Big(\sqrt{1 - \tilde{\lambda}\partial_1^2} - 1\Big) f - \frac{1}{4} g(f^2+g^2)^2\Big(\sqrt{1 - \tilde{\lambda}\partial_1^2} - 1\Big) g + \\ & + \frac{3\tilde{\lambda}^2}{4}\Big[2(f'^2+g'^2)(ff'+gg')^2 - (f^2+g^2)(f'^2+g'^2)^2\Big] + \\ & + \frac{\tilde{\lambda}^3}{2}\Big[2\mathbf{c}(f'^2+g'^2)^3 - \mathbf{b}(f'f''+g'g'') \times \\ & \qquad \times \Big((f^2+g^2)(f'f''+g'g'') - (f'^2+g'^2)(ff'+gg') - \\ & \qquad\quad - 2(ff'+gg')(ff''+gg'')\Big) + \\ & \qquad + \mathbf{a}(f'^2+g'^2)\Big(2(f'^2+g'^2)^2 + (ff''+gg'')^2 + \\ & \qquad\qquad + 3(f'^2+g'^2)(ff''+gg'') - \\ & \qquad\qquad - 2(f'f''+g'g'')(ff'+gg') - (f^2+g^2)(f''^2+g''^2)\Big) + \\ & \qquad + \mathbf{a}(ff'+gg')^2(f''^2+g''^2)\Big]\bigg\} + O\Big(\frac{\tilde{\lambda}^4}{J^2}\Big) \,. \end{aligned} \tag{3.6}$$

Let us first consider the quadratic approximation. The linearized equations of motion for the fluctuations are

$$\dot{f} = -\Big(1 - \sqrt{1 - \tilde{\lambda}\partial_1^2}\Big) g \,, \qquad \dot{g} = (1 - \sqrt{1 - \tilde{\lambda}\partial_1^2}) f \,, \tag{3.7}$$

and their solution may be written as

$$f(t,\sigma) = \frac{1}{2}\sum_{n=-\infty}^{\infty} (a_n e^{-i\omega_n t + in\sigma} + a_n^\dagger e^{i\omega_n t - in\sigma}) \,, \qquad \omega_n = \sqrt{1 + \tilde{\lambda} n^2} - 1 \,, \tag{3.8}$$

$$g(t,\sigma) = \frac{1}{2}\sum_{n=-\infty}^{\infty}(-ia_n e^{-i\omega_n t + in\sigma} + ia_n^\dagger e^{i\omega_n t - in\sigma}), \tag{3.9}$$

for real f and g. Upon quantization (3.7) becomes the equations of motion for the operators f, g

$$\dot{f} = i[\bar{H}_2, f], \qquad \dot{g} = i[\bar{H}_2, g], \qquad \bar{H}_2 \equiv \int_0^{2\pi}\frac{d\sigma}{2\pi}H_2, \tag{3.10}$$

provided we use the canonical commutation relations in (2.11)

$$[f(t,\sigma), f(t,\sigma')] = 0, \qquad [g(t,\sigma), g(t,\sigma')] = 0, \qquad [f(t,\sigma), g(t,\sigma')] = i\pi\delta(\sigma-\sigma'). \tag{3.11}$$

Then the coefficients in (3.8), (3.9) satisfy

$$[a_n, a_m^\dagger] = \delta_{n-m}, \tag{3.12}$$

so that a_n and $a_n^\dagger$ can be interpreted as annihilation and creation operators, with the vacuum state $|0\rangle$ defined by $a_n|0\rangle = 0$, for all integer n. A general oscillator state is

$$|\Psi\rangle = \prod_{n=-\infty}^{\infty}\frac{(a_n^\dagger)^{k_n}}{\sqrt{k_n!}}|0\rangle. \tag{3.13}$$

The integrated hamiltonian $\bar{H}_2$ then becomes

$$\bar{H}_2 = \sum_{n=-\infty}^{\infty} \omega_n \, a_n^\dagger a_n, \tag{3.14}$$

where we have used the normal ordering to ensure that the vacuum energy is zero, since the BMN vacuum is a BPS state in both gauge theory and string theory.

One also needs to impose the extra constraint that the total σ-momentum is zero, which gives [17]

$$\sum_{n=-\infty}^{\infty} n a_n^\dagger a_n = 0. \tag{3.15}$$

Then for physical oscillator states we get

$$\sum_{n=-\infty}^{\infty} n a_n^\dagger a_n |\Psi\rangle = 0, \qquad \sum_{n=-\infty}^{\infty} n k_n = 0. \tag{3.16}$$

Below we shall consider the "M-impurity" states as oscillator states with $k_n = 1$:

$$|M\rangle = a_{n_1}^\dagger \ldots a_{n_M}^\dagger |0\rangle, \tag{3.17}$$

where for simplicity we shall assume that all n_j are different (generalization to states with several equal n_j is straightforward, at least for $1/J$ corrections). Then the zero-momentum condition (3.16) gives

$$\sum_{j=1}^{M} n_j = 0, \tag{3.18}$$

and the leading term in the energy of an M-impurity state takes the familiar form [30, 1]

$$E^{(0)} = \langle M|\bar{H}_2|M\rangle = \sum_{j=1}^{M} (\sqrt{1+\tilde{\lambda} n_j^2} - 1) . \tag{3.19}$$

It is useful to make a comment on the choice of parameters. In the LL approach we use $J = J_1 + J_2$ as a natural total angular momentum, corresponding to a "fast" collective coordinate. Here M is a characteristic of a particular state, while it is J that enters into the background-independent form of the LL action (2.1). This is in line with gauge/spin chain intuition, where the use of total J or spin chain length as the state-independent parameter is natural. At the same time, on the string side, when expanding near a BPS state, i.e. a massless geodesic with spin J_1, one builds up J_2 from quantum excitations, and here it is natural to use $J' = J_1$ and $M = J_2$ as the basic parameters of, respectively, the vacuum and the state. Thus, compared to generic states in the SU(2) sector that carry spins (J_1, J_2) with $J = J_1 + J_2$, here we have [17]

$$J_1 = J - M , \qquad J_2 = M . \tag{3.20}$$

The corresponding gauge-theory states are $\mathrm{Tr}(\Phi_1^{J_1} \Phi_2^{J_2}) + \cdots$, and J plays the role of the length of the spin chain and M is the number of magnons. On the string side, the LL approach is adapted to semiclassical solutions for which J_1 is of order J_2 rather than to near-BMN states which are small fluctuations near the vacuum and for which $J_1 \gg J_2$. In describing BMN states in the LL approach one has an "unnatural" choice of parameters: $\tilde{\lambda} \equiv \lambda/J^2$, not the usual BMN effective coupling $\lambda' \equiv \lambda/J'^2$. In the LL description the BMN energy E starts with J to which we add terms of order $\tilde{\lambda}$, i.e. $E = J + E^{(0)} + O(\frac{1}{J}) = J + \sum_{j=1}^{M} (\sqrt{1+\tilde{\lambda} n_j^2} - 1) + O(\frac{1}{J})$, while the equivalent string theory expression is $E = J' + \sum_{j=1}^{M} \sqrt{1+\lambda' n_j^2} + O(\frac{1}{J'})$.

4. $1/J$ corrections to the BMN spectrum

Let us now generalize the computation of the $1/J$ corrections to the energy (3.19) in [17] to $\tilde{\lambda}^3$ order. To compute the $1/J$ correction to the energy of an M-impurity state one needs to include the quartic term in the hamiltonian (3.5) integrated over σ, i.e. $\bar{H}_4 \equiv \int_0^{2\pi} \frac{d\sigma}{2\pi} H_4$, and use the standard quantum mechanical perturbation theory. Written in terms of the creation and annihilation operators, $\bar{H}_4$ is found to be

$$\begin{aligned} \bar{H}_4 = \frac{1}{J} \sum_{n,m,k,l} \left[c_{nmk} + \frac{3\tilde{\lambda}^2}{32} nmkl + \frac{\tilde{\lambda}^3}{16} \left(\mathbf{a}\, nmk^2 l^2 - \mathbf{b}\, nm^2 kl^2 \right) + O(\tilde{\lambda}^4) \right] \times \\ \times \left(a_n a_m^\dagger a_k a_l^\dagger + a_n a_m^\dagger a_{-k}^\dagger a_{-l} + a_n^\dagger a_m a_{-k} a_{-l}^\dagger + a_n^\dagger a_m a_k^\dagger a_l \right) \delta_{n-m+k-l} , \end{aligned} \tag{4.1}$$

where

$$c_{nmk} \equiv \frac{1}{4}\sqrt{1+\tilde{\lambda}(n-m)^2} - \frac{1}{8}\sqrt{1+\tilde{\lambda} n^2} - \frac{1}{8}\sqrt{1+\tilde{\lambda}(n-m+k)^2} . \tag{4.2}$$

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In the expression for the interacting hamiltonian we have dropped the time dependent phases ($e^{-i\omega_n t}$) since they can be removed by a unitary transformation with the quadratic hamiltonian $\bar{H}_2$. Here and in what follows the summations over n, m, *etc.*, are from $-\infty$ to ∞.

As discussed in [17], to obtain the results consistent with both the gauge-theory spin chain and the string-theory expressions one should use a normal ordering prescription for $\bar{H}_4$. Doing so we get

$$\bar{H}_4 = \frac{1}{J}\sum_{n,m} h_{nm}\, a_n^\dagger a_m^\dagger a_n a_m\,, \tag{4.3}$$

$$\begin{aligned} h_{nm} &= 1 + \sqrt{1+\tilde{\lambda}(n-m)^2} - 2\sqrt{1+\tilde{\lambda}n^2} + \\ &\quad + \frac{3\tilde{\lambda}^2}{4}n^2m^2 + \frac{\tilde{\lambda}^3}{16}n^2m^2[2\mathbf{a}(n+m)^2 + \mathbf{b}(n-m)^2] + O(\tilde{\lambda}^4)\,. \end{aligned} \tag{4.4}$$

Then the leading correction to the energy (3.19) of an M-impurity state is given by

$$\begin{aligned} \langle M|\bar{H}_4|M\rangle = \frac{1}{J}\Bigg\{ & M^2 - 2M + \sum_{i,j=1}^{M}\sqrt{1+\tilde{\lambda}(n_i-n_j)^2} - 2(M-1)\sum_{i=1}^{M}\sqrt{1+\tilde{\lambda}n_i^2} + \\ & + \frac{3\tilde{\lambda}^2}{4}\sum_{i=1}^{M}n_i^4 + \frac{\tilde{\lambda}^3}{16}\Bigg[-8\mathbf{a}\sum_{i=1}^{M}n_i^6 + \sum_{i,j=1}^{M}n_i^2n_j^2\Big[2\mathbf{a}(n_i+n_j)^2 + \\ & + \mathbf{b}(n_i-n_j)^2\Big]\Bigg] + O(\tilde{\lambda}^4)\Bigg\}\,. \end{aligned} \tag{4.5}$$

Expanding in $\tilde{\lambda}$ gives for the $1/J$ correction to the energy

$$\begin{aligned} E^{(1)} = \langle M|\bar{H}_4|M\rangle = \frac{1}{J}\Bigg\{ & \tilde{\lambda}\sum_{i=1}^{M}n_i^2 - \tilde{\lambda}^2\sum_{i=1}^{M}n_i^4 + \frac{1}{8}(1-4\mathbf{a})\tilde{\lambda}^3\sum_{i=1}^{M}n_i^6 + \\ & + \frac{\tilde{\lambda}^3}{16}\sum_{i,j=1}^{M}n_i^2n_j^2\Big[(2\mathbf{a}+\mathbf{b}+15)(n_i^2+n_j^2) + 2(2\mathbf{a}-\mathbf{b}-10)n_in_j\Big] + \\ & + O(\tilde{\lambda}^4)\Bigg\} \end{aligned} \tag{4.6}$$

Plugging in the string-theory and gauge-theory coefficients $\mathbf{a}, \mathbf{b}$ in (2.3), (2.4) we conclude that this expression is in precise agreement with the full string theory computation in [5, 31] and with the result found using the gauge and string Bethe ansätze in [24], expanded up to $\tilde{\lambda}^4/J$ order. This agreement confirms, in particular, the values of the coefficients $\mathbf{a}$ and $\mathbf{b}$ in the gauge-theory LL action given in (2.4) (see also appendix A).

Let us recall again that in comparing with the near-BMN results of [24, 31], one should note that J as defined there is J_1 in the SU(2) sector notation. To compare with our results, one may define $J' \equiv J_1 = J - M$ and $\lambda' = \lambda/J'^2$; the expressions of [24, 31] should have $(J, \tilde{\lambda})$ replaced with (J', λ') and then re-expressed in terms of the parameters $(J, \tilde{\lambda})$ which are natural in the present LL approach.

The difference between the order $\tilde{\lambda}^3/J$ string and gauge theory corrections to the BMN energy is because of the difference of the values of the coefficient $\mathbf{b}$: $\mathbf{b}_s - \mathbf{b}_g = -1$. The energy difference is thus [4]

$$E_s^{(1)} - E_g^{(1)} = -\frac{\tilde{\lambda}^3}{16J}\sum_{i,j=1}^{M} n_i^2 n_j^2 (n_i - n_j)^2 + O\left(\frac{\tilde{\lambda}^4}{J}\right). \tag{4.7}$$

We shall return to the discussion of this difference in section 6.

In the string case the last double-sum term in (4.6) is $-\frac{\tilde{\lambda}^3}{16}\sum_{i,j=1}^{M} n_i^2 n_j^2 (n_i + n_j)^2$ while in the gauge case it is $-\frac{\tilde{\lambda}^3}{4}\sum_{i,j=1}^{M} n_i^3 n_j^3$. As discussed in appendix B, the gauge-theory expression in (4.6) has a simple spin-chain generalization to all orders in $\tilde{\lambda}$ implied by the gauge Bethe ansatz [14, 24][5]

$$E_g^{(1)} = \frac{\tilde{\lambda}}{J}\left(\sum_{i=1}^{M}\frac{n_i^2}{1+\tilde{\lambda}n_i^2} - \sum_{i,j=1}^{M}\frac{n_i n_j}{\sqrt{1+\tilde{\lambda}n_i^2}\,\sqrt{1+\tilde{\lambda}n_i^2}}\right). \tag{4.8}$$

5. $1/J^2$ corrections to BMN spectrum

To find $1/J^2$ corrections we follow the method in [17] where order $\tilde{\lambda}/J^2$ terms were computed. We need to combine the second order perturbation theory correction for the quartic hamiltonian (3.5) with the first order perturbation theory correction for the sixth order hamiltonian in (3.6). The regularization issues were discussed in detail in [17]: to match string/gauge results we should use the normal-ordered form of the hamiltonians and apply ζ-function regularization for intermediate-state sums. We shall also need to add a local higher-derivative "counterterm" which (on gauge side) is a lattice correction to the continuum limit of the LL action (see [17] and below).

5.1 Second-order perturbation ("exchange") contribution

Starting with the quartic hamiltonian (4.1) we need to compute

$$\langle M|(\bar{H}_4)^{(2)}|M\rangle = \sum_{M\neq M'}\frac{\langle M|\bar{H}_4|M'\rangle\langle M'|\bar{H}_4|M\rangle}{E_M - E_{M'}}, \tag{5.1}$$

where $|M'\rangle$ is any possible intermediate state, and $|M\rangle = a^\dagger_{n_1}\dots a^\dagger_{n_M}|0\rangle$. Since $\bar{H}_4$ in (4.3) contains only terms of the form $(a^\dagger)^2 a^2$, the only non-trivial intermediate states can be the $M' = M$ -particle states of the form $a^\dagger_{n'_1}\dots a^\dagger_{n'_M}|0\rangle$. Then in order for the matrix element $\langle 0|a_{n_1}\dots a_{n_M}|\bar{H}_4|a^\dagger_{n'_1}\dots a^\dagger_{n'_M}|0\rangle$ to be non-zero, there should be a j and k such that $n'_j = n_j + q$ and $n'_k = n_k - q$, with all other $n'_i = n_i$, $i \neq j,k$. In order for $|M\rangle$ to be distinct from $|M'\rangle$, we require that $0 \neq q \neq n_k - n_j$. With these conditions, we then find

[5] This is equivalent to eq. (3.8) in [24] after observing that J there is $J-M$ here and λ' there is $\frac{\lambda}{(J-M)^2}$ here. As always, we assume the condition (3.18).

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that if $n_k \neq n_j$

$$
\begin{aligned}
\mathrm{E}_4^{(1)} \equiv \langle M|\bar{H}_4|M'\rangle = \frac{1}{J}\Bigg\{ & 2\sqrt{1+\tilde{\lambda}q^2} + 2\sqrt{1+\tilde{\lambda}(n_k-n_j-q)^2} - \sqrt{1+\tilde{\lambda}n_k^2} - \sqrt{1+\tilde{\lambda}n_j^2} - \\
& - \sqrt{1+\tilde{\lambda}(q+n_j)^2} - \sqrt{1+\tilde{\lambda}(n_k-q)^2} + \\
& + \frac{3\tilde{\lambda}^2}{2} n_k n_j (q+n_j)(n_k-q)\Bigg[1 + \frac{\tilde{\lambda}}{12}\Bigg(2\mathbf{a}(n_j+n_k)^2 + \\
& \qquad + \mathbf{b}\Big[(n_k-n_j)^2 - \\
& \qquad\qquad - 2q(n_k-n_j-q)\Big]\Bigg)\Bigg]\Bigg\}, \qquad (5.2)
\end{aligned}
$$

where $n_j + q$ and $n_k - q$ are not equal to one of the other n_l's. The energy difference in (5.1) is

$$
W_1 \equiv E_M - E_{M'} = \sqrt{1+\tilde{\lambda}n_j^2} + \sqrt{1+\tilde{\lambda}n_k^2} - \sqrt{1+\tilde{\lambda}(n_j+q)^2} - \sqrt{1+\tilde{\lambda}(n_k-q)^2}\,. \qquad (5.3)
$$

If $n_j + q = n_l$, and so $|M'\rangle$ has two impurities with the same momenta, then the matrix element is

$$
\begin{aligned}
\mathrm{E}_4^{(2)} \equiv \langle M|\bar{H}_4|M'\rangle = \frac{\sqrt{2}}{J}\Bigg\{ & 2\sqrt{1+\tilde{\lambda}(n_l-n_j)^2} + 2\sqrt{1+\tilde{\lambda}(n_k-n_l)^2} - \sqrt{1+\tilde{\lambda}n_k^2} - \\
& - \sqrt{1+\tilde{\lambda}n_j^2} - \sqrt{1+\tilde{\lambda}n_l^2} - \sqrt{1+\tilde{\lambda}(n_k-n_l+n_j)^2} \\
& + \frac{3\tilde{\lambda}^2}{2} n_k n_j n_l (n_k+n_j-n_l) \times \\
& \times \Bigg(1 + \frac{\tilde{\lambda}}{12}\Bigg[2\mathbf{a}(n_j+n_k)^2 + \mathbf{b}\Big[(n_k-n_j)^2 - \qquad (5.4) \\
& \qquad - 2(n_l-n_j)(n_k-n_l)\Big]\Bigg]\Bigg)\Bigg\},
\end{aligned}
$$

and the energy difference is

$$
W_2 \equiv E_M - E_{M'} = \sqrt{1+\tilde{\lambda}n_j^2} + \sqrt{1+\tilde{\lambda}n_k^2} - \sqrt{1+\tilde{\lambda}n_l^2} - \sqrt{1+\tilde{\lambda}(n_j+n_k-n_l)^2}\,. \qquad (5.5)
$$

Then the "exchange" contribution is given by

$$
\langle M|(\bar{H}_4)^{(2)}|M\rangle = \frac{1}{4}\sum_{\substack{j,k\\ j\neq k}}^{M}\left[\sum_{\substack{q=-\infty\\ 0\neq q\neq n_k-n_j}}^{\infty} \frac{(\mathrm{E}_4^{(1)})^2}{W_1} + \sum_{\substack{l\neq j\\ l\neq k}}^{M}\frac{(\mathrm{E}_4^{(2)})^2}{W_2}\right]. \qquad (5.6)
$$

Interpreted as a string-theory expression (i.e. non-perturbative in $\tilde{\lambda}$, with sums done before the expansion in $\tilde{\lambda}$), the sum over the "virtual" momentum q produces, as we shall see in section 6, a contribution which is non-analytic in $\tilde{\lambda}$. This is a novel phenomenon [4], which was absent at order $1/J$.

Let us first ignore such contributions and expand in $\tilde{\lambda}$ before doing the sum over q. (This is the procedure which is in any case appropriate for the gauge-theory.) This will give us terms which are analytic in $\tilde{\lambda}$. It was shown in [17] that the quantum LL hamiltonian and the gauge theory Bethe ansatz give the same $\tilde{\lambda}/J^2$ correction (the exact superstring result is not yet available). Our aim here is to extend the computation of [17] to the orders $\tilde{\lambda}^2/J^2$ and $\tilde{\lambda}^3/J^2$ and to show that the results match the gauge and string Bethe ansatz results found in appendix B.

For the part of $\tilde{\lambda}^2/J^2$ correction coming from the "exchange" contribution (5.1), (5.6) we obtain

$$\frac{\tilde{\lambda}^2}{J^2}\left[\sum_{\substack{k,j=1\\k\neq j}}^{M}\frac{n_j^3 n_k^3}{(n_k-n_j)^2}+\frac{9}{8}(M-4)\left(\sum_{i=1}^{M}n_i^2\right)^2+\frac{1}{8}(M^2-8M+20)\sum_{i=1}^{M}n_i^4\right]. \tag{5.7}$$

We see that the pole-type contribution $\sim \frac{1}{(n_k-n_j)^2}$ matches the corresponding term in the correction to the BMN energy (B.14) computed from the Bethe ansatz. To compare other "local" terms we must add the "contact" contribution coming from the expectation value of the sixth-order hamiltonian and also add a local counterterm [17] contribution (see below). Note that at orders $\tilde{\lambda}$ and $\tilde{\lambda}^2$ the string and gauge Bethe ansätze produce the same expressions, so they both agree with what we find from the LL approach.

Next, let us consider the order $\tilde{\lambda}^3/J^2$ term. We first concentrate on finding the pole term $\sim \frac{1}{(n_k-n_j)^2}$ in the first line of (B.14). It can come only from the first term in (5.6), where the order $\tilde{\lambda}^3/J^2$ part is

$$\frac{(\mathrm{E}_4^{(1)})^2}{W_1} \to \frac{\tilde{\lambda}^3}{16J^2}\frac{1}{v_{kj}}\Big[A_1+v_{kj}(A_2+v_{kj}[A_3-v_{kj}(A_4-2v_{kj})])\Big], \tag{5.8}$$

$$v_{kj} \equiv q(n_k-n_j-q). \tag{5.9}$$

Here A_1, A_2, A_3, A_4 are polynomials in n_j which do not depend on q. The pole term comes from (see also [17])

$$\begin{aligned}\frac{1}{4}\sum_{\substack{j,k\\j\neq k}}\sum_{q}\frac{(\mathrm{E}_4^{(1)})^2}{W_1} &\to \sum_{\substack{k,j=1\\k\neq j}}^{M}\sum_{\substack{q=-\infty\\0\neq q\neq n_k-n_j}}^{\infty}\frac{A_1}{64q(n_k-n_j-q)}\\ &= -\frac{1}{32}\sum_{\substack{k,j=1\\k\neq j}}^{M}\frac{A_1}{(n_k-n_j)^2}\\ &= \frac{1}{4}\mathbf{b}\sum_{\substack{k,j=1\\k\neq j}}^{M}n_j^3n_k^3-\frac{1}{4}\sum_{\substack{k,j=1\\k\neq j}}^{M}\frac{n_j^2n_k^2}{(n_k-n_j)^2}\Big[3n_j^4+3n_k^4-18n_j^3n_k-18n_jn_k^3+\\ &\qquad +19n_j^2n_k^2-2\mathbf{a}\,n_jn_k(n_j+n_k)^2\Big].\end{aligned} \tag{5.10}$$

We observe that the pole term depends only on $\mathbf{a}$, which is the same for the gauge and string LL actions. Setting $\mathbf{a} = -7/4$ as in (2.3), (2.4) we get for the pole term

$$-\frac{3\tilde{\lambda}^3}{4J^2}\sum_{\substack{k,j=1\\k\neq j}}^{M}\frac{n_j^4 n_k^4}{(n_k-n_j)^2}\,. \tag{5.11}$$

This indeed matches the expression for the corresponding pole term in both the gauge and string Bethe ansatz results (B.14) and (B.23).

Computing the sums in (5.6) with all other "local" terms included we find the following total result for the $\tilde{\lambda}^3/J^2$ contribution from the second order perturbation theory correction[6]

$$\begin{aligned}
E^{(2)}_{3\ \text{exch.}} = \frac{\tilde{\lambda}^3}{48J^2}\Bigg[&(113+56\mathbf{a})\left(\sum_{j=1}^{M}n_j^2\right)^3 - 6(116+8\mathbf{b}-5M)\left(\sum_{j=1}^{M}n_j^3\right)^2 + \\
&+(304+136\mathbf{a}+36\mathbf{b}+96M-3M^2)\sum_{j=1}^{M}n_j^6 + \\
&+(426-192\mathbf{a}+24\mathbf{b}-135M)\sum_{i=1}^{M}n_i^4\sum_{j=1}^{M}n_j^2 + \\
&+12(8\mathbf{a}+11)\sum_{i\neq j}^{M}\frac{n_i^4 n_j^4}{(n_i-n_j)^2}\Bigg]\,.
\end{aligned} \tag{5.12}$$

5.2 Sixth-order "contact" contribution

Let us now compute the contribution coming from the expectation value of the sixth order term in the LL hamiltonian (3.6). After a lengthy computation we obtain the following expression for the normal ordered form for this term[7]

$$\bar{H}_6 = \frac{1}{J^2}\sum_{n,m,k} h_{nmk}\, a_n^\dagger a_m^\dagger a_k^\dagger a_n a_m a_k\,, \tag{5.13}$$

$$\begin{aligned}
h_{nmk} = \frac{1}{4}\Big[&\sqrt{1+\tilde{\lambda}(n-m+k)^2}+\sqrt{1+\tilde{\lambda}(n-m-k)^2}-2\sqrt{1+\tilde{\lambda}n^2}\Big] + \\
&+\frac{1}{2}n^2mk\Big\{-9\tilde{\lambda}^2+\tilde{\lambda}^3\Big[(9\mathbf{a}+12\mathbf{c})mk-(\mathbf{a}+\mathbf{b})n^2+(\mathbf{b}-14\,\mathbf{a})nm\Big]\Big\}+O(\tilde{\lambda}^4)\,.
\end{aligned} \tag{5.14}$$

Expanding the square roots in h_{nmk} in powers of $\tilde{\lambda}$ we can find contributions to the $1/J^2$ correction to order $\tilde{\lambda}^3$. At order $\tilde{\lambda}$ we already computed the resulting expectation value in [17]. At order $\tilde{\lambda}^2$ we have

$$h^{(2)}_{mnk} = -\frac{\tilde{\lambda}^2}{8J^2}(n^4+9n^2m^2-4n^3m+30n^2mk)\,. \tag{5.15}$$

[6] As in [17] here we used the ζ-function regularization, i.e. set $\sum_{n=-\infty}^{\infty} n^s = 0$ for $s = 0,1,2,\ldots$

[7] In this subsection the only regularization we use is the assumption that the sixth order term in the hamiltonian is normal ordered.

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For the expectation vales in an M-impurity state we get

$$\langle M|\sum_{n,m,k} n^4 a_n^\dagger a_m^\dagger a_k^\dagger a_n a_m a_k|M\rangle = (M-1)(M-2)\sum_{j=1}^{M} n_j^4\,, \tag{5.16}$$

$$\langle M|\sum_{n,m,k} n^2m^2 a_n^\dagger a_m^\dagger a_k^\dagger a_n a_m a_k|M\rangle = (M-2)\left(\sum_{j=1}^{M} n_j^2\right)^2 - (M-2)\sum_{j=1}^{M} n_j^4\,, \tag{5.17}$$

$$\langle M|\sum_{n,m,k} n^3m\, a_n^\dagger a_m^\dagger a_k^\dagger a_n a_m a_k|M\rangle = -(M-2)\sum_{j=1}^{M} n_j^4\,, \tag{5.18}$$

$$\langle M|\sum_{n,m,k} n^2mk\, a_n^\dagger a_m^\dagger a_k^\dagger a_n a_m a_k|M\rangle = -\left(\sum_{j=1}^{M} n_j^2\right)^2 + 2\sum_{j=1}^{M} n_j^4\,. \tag{5.19}$$

Then the $\tilde\lambda^2$ contribution is found to be

$$E^{(2)}_{2\ \text{cont.}} = -\frac{\tilde\lambda^2}{8J^2}\left[(9M-48)\left(\sum_{j=1}^{M} n_j^2\right)^2 + (M^2-8M+72)\sum_{j=1}^{M} n_j^4\right]. \tag{5.20}$$

Adding it to the corresponding exchange term (5.7) we observe that all M-dependent coefficients cancel and the result is

$$E^{(2)}_{2\ \text{exch.+cont.}} = \frac{\tilde\lambda^2}{J^2}\left[\frac{3}{2}\left(\sum_{j=1}^{M} n_j^2\right)^2 - \frac{13}{2}\sum_{j=1}^{M} n_j^4 + \sum_{\substack{k,j=1\\ k\neq j}}^{M} \frac{n_j^3 n_k^3}{(n_k-n_j)^2}\right]. \tag{5.21}$$

The $\tilde\lambda^3$ term in h_{mnk} is

$$\begin{aligned} h^{(3)}_{mnk} = \frac{\tilde\lambda^3}{16J^2}\Big[&n^6 + 45n^4m^2 - (15+8\,\mathbf{a}+8\,\mathbf{b})n^4mk - 10n^3m^3 - \\ &- (60+112\,\mathbf{a}-8\,\mathbf{b})n^3m^2k - 6n^5m + (45+72\,\mathbf{a}+96\,\mathbf{c})n^2m^2k^2\Big]. \end{aligned} \tag{5.22}$$

For M-impurity states we get

$$\langle M|\sum_{n,m,k} n^6 a_n^\dagger a_m^\dagger a_k^\dagger a_n a_m a_k|M\rangle = (M-2)(M-1)\sum_{j=1}^{M} n_j^6\,, \tag{5.23}$$

$$\langle M|\sum_{n,m,k} n^4m^2 a_n^\dagger a_m^\dagger a_k^\dagger a_n a_m a_k|M\rangle = (M-2)\sum_{j=1}^{M} n_j^4\sum_{k=1}^{M} n_k^2 - (M-2)\sum_{j=1}^{M} n_j^6\,, \tag{5.24}$$

$$\langle M|\sum_{n,m,k} n^4mk a_n^\dagger a_m^\dagger a_k^\dagger a_n a_m a_k|M\rangle = -\sum_{j=1}^{M} n_j^4\sum_{k=1}^{M} n_k^2 + 2\sum_{j=1}^{M} n_j^6\,, \tag{5.25}$$

$$\langle M|\sum_{n,m,k} n^3m^2k\, a_n^\dagger a_m^\dagger a_k^\dagger a_n a_m a_k|M\rangle = -\left(\sum_{j=1}^{M} n_j^3\right)^2 - \sum_{j=1}^{M} n_j^4\sum_{k=1}^{M} n_k^2 + 2\sum_{j=1}^{M} n_j^6\,, \tag{5.26}$$

$$\langle M|\sum_{n,m,k} n^5m a_n^\dagger a_m^\dagger a_k^\dagger a_n a_m a_k|M\rangle = -(M-2)\sum_{j=1}^{M} n_j^6\,, \tag{5.27}$$

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$$\langle M| \sum_{n,m,k} n^2 m^2 k^2 a_n^\dagger a_m^\dagger a_k^\dagger a_n a_m a_k |M\rangle = \left(\sum_{j=1}^M n_j^2\right)^3 - 3\sum_{j=1}^M n_j^4 \sum_{k=1}^M n_k^2 + 2\sum_{j=1}^M n_j^6 \,, \quad (5.28)$$

$$\langle M| \sum_{n,m,k} n^3 m^3 a_n^\dagger a_m^\dagger a_k^\dagger a_n a_m a_k |M\rangle = (M-2)\left(\sum_{j=1}^M n_j^3\right)^2 - (M-2)\sum_{j=1}^M n_j^6 \,. \quad (5.29)$$

Then the contact contribution from (5.22) combined with the exchange contribution (5.12) from the second order perturbation theory gives the following $\tilde{\lambda}^3/J^2$ result

$$\begin{aligned} E^{(2)}_{3\ \text{exch.+cont.}} = \frac{\tilde{\lambda}^3}{12J^2}\Bigg[&(62 + 68\mathbf{a} + 72\mathbf{c})\left(\sum_{j=1}^M n_j^2\right)^3 + 6(-19 + 14\mathbf{a} - 3\mathbf{b})\left(\sum_{j=1}^M n_j^3\right)^2 + \\ &+ (76 - 38\mathbf{a} + 9\mathbf{b} + 144\mathbf{c})\sum_{j=1}^M n_j^6 + \\ &+ (-6 - 120\mathbf{a} + 6\mathbf{b} - 216\mathbf{c})\sum_{j=1}^M n_j^4 \sum_{i=1}^M n_i^2 + \\ &+ (24\mathbf{a} + 33)\sum_{i\neq j}^M \frac{n_i^4 n_j^4}{(n_i - n_j)^2}\Bigg] \,. \end{aligned} \quad (5.30)$$

The final result for the $1/J^2$ correction is found after adding a higher-derivative term contribution discussed in the next subsection.

5.3 Contribution of higher-derivative counterterms

As was already discussed in [17], when approximating the discrete spin chain coherent state action by the continuous LL action one drops certain higher-derivative corrections. These are suppressed by $1/J$ factors in the classical large J limit, but they need to be re-instated to correctly reproduce the spin chain result for the $1/J^n$ contributions in the LL approach. These terms should also be present in the full string-theory result, where they should originate from contributions of other modes outside the LL subsector (one may view the string LL action as a result of integrating out all other superstring world-sheet fields while keeping $\vec{n}$ as a background).

The relevant higher-derivative terms in the LL action can be obtained, e.g., as follows. The local part of all-order dilatation operator which contributes only to terms quadratic in $\vec{n}$ can be represented in the following symbolic way that correctly captures the combinatorics of the expansion [28]

$$D \approx \sum_{l=1}^J \left(\sqrt{1 + 2D_{l,l+1}} - 1\right) . \quad (5.31)$$

Here $D_{l,l+1} = I - P_{l,l+1}$ is the "density" of the one-loop dilatation operator. As usual, to find the LL lagrangian [19, 28] one should take the coherent state expectation value, and then the continuum limit. The approximate equality in (5.31) means that expanding the square root expression and taking the coherent state expectation value correctly reproduces the leading order $\vec{n}^2$ term in the resulting hamiltonian. In the continuum limit

$$\vec{n}_{l+1} - \vec{n}_l = \frac{2\pi}{J}\partial_1\vec{n} + \frac{1}{2}\left(\frac{2\pi}{J}\right)^2 \partial_1^2\vec{n} + \frac{1}{6}\left(\frac{2\pi}{J}\right)^3 \partial_1^3\vec{n} + \cdots \,, \tag{5.32}$$

and $\sum_{l=1}^{J} \to J\int_0^{2\pi}\frac{d\sigma}{2\pi}$. Ignoring total derivative terms we have

$$\langle n|D_{l,l+1}|n\rangle = \frac{\lambda}{2(4\pi)^2}(\vec{n}_{l+1} - \vec{n}_l)^2 \;\to\; -\frac{\lambda}{8J^2}\vec{n}\Big(\partial_1^2 + \frac{\pi^2}{3J^2}\partial_1^4 + \cdots\Big)\vec{n}\,. \tag{5.33}$$

Then the relevant part of the coherent state expectation value of D may be written as [28]

$$\langle n|D|n\rangle = J\int_0^{2\pi}\frac{d\sigma}{2\pi}\frac{1}{4}\vec{n}\left[\sqrt{1 - \tilde{\lambda}\Big(\partial_1^2 + \frac{\pi^2}{3J^2}\partial_1^4 + \cdots\Big)} - 1\right]\vec{n} + O(\vec{n}^4)\,. \tag{5.34}$$

Expanding for large J one finds

$$\langle n|D|n\rangle = J\int_0^{2\pi}\frac{d\sigma}{2\pi}\left[\frac{1}{4}\vec{n}\left(\sqrt{1-\tilde{\lambda}\partial_1^2} - 1\right)\vec{n} - \frac{\tilde{\lambda}\pi^2}{24J^2}\vec{n}\frac{1}{\sqrt{1-\tilde{\lambda}\partial_1^2}}\partial_1^4\vec{n} + \cdots\right]. \tag{5.35}$$

The leading term here is the first term in the LL hamiltonian in (2.2). To obtain the full $1/J^2$ correction one needs to keep the next-order term in the above expansion (5.35) or add the following term to the LL lagrangian

$$\Delta L_2 = \frac{\tilde{\lambda}\pi^2}{24J^2}\;\vec{n}\frac{1}{\sqrt{1-\tilde{\lambda}\partial_1^2}}\partial_1^4\vec{n} + O\Big(\frac{1}{J^4}\Big)\,. \tag{5.36}$$

Using the momentum representation for fluctuations of $\vec{n}$ (i.e. $f, g \sim e^{in\sigma}$), and noticing that to quadratic order one can make the replacement $\partial_1^2 \to -n^2$, we obtain the additional $1/J^2$ correction to the BMN energy

$$\langle M|\Delta\bar{H}_2|M\rangle = -\frac{\tilde{\lambda}\pi^2}{6J^2}\sum_{i=1}^{M}\frac{n_i^4}{\sqrt{1+\tilde{\lambda}n_i^2}} + O\Big(\frac{1}{J^4}\Big)\,. \tag{5.37}$$

This expression matches the first term in the corresponding energy (B.13) computed from the Bethe ansatz, both on the gauge and the string side. Explicitly expanding in powers of $\tilde{\lambda}$ we get

$$\langle M|\Delta\bar{H}_2|M\rangle = -\frac{\tilde{\lambda}\pi^2}{6J^2}\sum_{j=1}^{M}n_j^4 + \frac{\tilde{\lambda}^2\pi^2}{12J^2}\sum_{j=1}^{M}n_j^6 - \frac{\tilde{\lambda}^3\pi^2}{16J^2}\sum_{j=1}^{M}n_j^8 + O\Big(\frac{\tilde{\lambda}^4}{J^2}\Big)\,. \tag{5.38}$$

5.4 Final results and discussion

We are now in a position to present the final results for the $1/J^2$ corrections obtained from the quantum LL hamiltonian, and to compare them with the corresponding expressions obtained in appendix B from the Bethe ansatz for the gauge theory [14] and for the string theory [24].

As was already mentioned, the order $\tilde{\lambda}/J^2$ correction was previously found from the quantum LL approach in [17] and was shown to be the same as the one following from the Bethe ansatz. Combining the results from (5.21) and (5.38) we obtain for the $\frac{\tilde{\lambda}^2}{J^2}$ correction

$$E_2^{(2)} = \frac{\tilde{\lambda}^2}{J^2}\left[\frac{\pi^2}{12}\sum_i^M n_i^6 - \frac{13}{2}\sum_i^M n_i^4 + \frac{3}{2}\left(\sum_i^M n_i^2\right)^2 + \sum_{i\neq j}^M \frac{n_i^3 n_j^3}{(n_i - n_j)^2}\right]. \tag{5.39}$$

It is, indeed, the same as the $\tilde{\lambda}^2/J^2$ term found from both gauge and string Bethe ansätze in appendix B.

Summing up (5.38) with (5.30) we obtain the total expression for the $\frac{\tilde{\lambda}^3}{J^2}$ term. Explicitly, for the gauge-theory values of the parameters $\mathbf{a}_g$, $\mathbf{b}_g$, $\mathbf{c}_g$ in (2.4) we get

$$\begin{aligned} E_{3g}^{(2)} &= \frac{\tilde{\lambda}^3}{J^2}\Bigg[-\frac{\pi^2}{16}\sum_i^M n_i^8 + \frac{49}{4}\sum_i^M n_i^6 - \frac{9}{4}\sum_i^M n_i^4 \sum_j^M n_j^2 - \frac{9}{2}\left(\sum_i^M n_i^3\right)^2 - \\ &\quad - \frac{1}{4}\left(\sum_i^M n_i^2\right)^3 - \frac{3}{4}\sum_{i\neq j}^M \frac{n_i^4 n_j^4}{(n_i - n_j)^2}\Bigg]. \end{aligned} \tag{5.40}$$

Remarkably, this is precisely the same as the result (B.14) found from the gauge-theory Bethe ansatz.

For the string values $\mathbf{a}_s$, $\mathbf{b}_s$, $\mathbf{c}_s$ in (2.3) we get

$$\begin{aligned} E_{3s}^{(2)} &= \frac{\tilde{\lambda}^3}{J^2}\Bigg[-\frac{\pi^2}{16}\sum_i^M n_i^8 + \frac{49}{4}\sum_i^M n_i^6 - \frac{31}{8}\sum_i^M n_i^4 \sum_j^M n_j^2 - 3\left(\sum_i^M n_i^3\right)^2 + \\ &\quad + \frac{1}{8}\left(\sum_i^M n_i^2\right)^3 - \frac{3}{4}\sum_{i\neq j}^M \frac{n_i^4 n_j^4}{(n_i - n_j)^2}\Bigg]. \end{aligned} \tag{5.41}$$

Again, this matches the expression (B.23) following from the string Bethe ansatz of [24].

The conclusion that the quantum LL approach reproduces the results from gauge theory Bethe ansatz is not totally surprising since the λ^3 coefficients in the gauge LL action were essentially derived from the gauge Bethe ansatz results. We believe our results are still interesting and non-trivial since they imply that various finite-size corrections can be systematically reproduced by quantizing a continuous effective action. Also, the matching serves as a self consistency check for the integrability of this system. At $\tilde{\lambda}^3$ order an integrable system is known that would reproduce the Bethe equations, the Inozemtsev chain [21] with couplings defined as in [6]. It would be very interesting to see if there is still matching at higher orders, since the spin chain that would give the Bethe equations in (B.1) and (B.2) is presently unknown.

The conclusion that the quantization of the "string" LL action (which is only a limit of the classical string action) leads to the same result for the $1/J^2$ corrections to the BMN energies as the string Bethe ansatz is more remarkable, since quantum integrability has not really been established at $\tilde{\lambda}^3$ order for this system, although some evidence of integrability has been presented in [33]. It also gives us certain confidence that the expression (5.41) is indeed the full $\tilde{\lambda}^3/J^2$ contribution that would follow from the direct superstring computation (which has not yet been carried out due to its complexity, cf. [32]). In addition, this provides a strong indication that the string Bethe ansatz of [24] does correctly describe the quantum $\tilde{\lambda}^3/J^2$ string-theory correction. At the same time, it also implies that it is only necessary to quantize the string modes that appear explicitly in the LL action which is relevant for large J in order to reproduce the exact quantum superstring results, provided one chooses a suitable regularization.

Our conclusions are also consistent with the suggestion [33] that the quantum string spectrum coming from the string Bethe ansatz in [24] can be found from a spin chain hamiltonian which has order λ^3 coefficients for the 4-spin terms that differ from those in the gauge-theory 3-loop dilatation operator of [22].[8]

As was first found in [5], the gauge and string order $\tilde{\lambda}^3/J$ terms in the near-BMN energy disagree. In general, for M impurities one finds from the Bethe ansatz expressions in [14, 24] a simple result (4.7) for the difference of the string and gauge energies which can also be written as

$$E_s^{(1)} - E_g^{(1)} = -\frac{\tilde{\lambda}^3}{8J}\left[\sum_i^M n_i^4 \sum_j^M n_j^2 - \left(\sum_i^M n_i^3\right)^2\right] + O\left(\frac{\tilde{\lambda}^4}{J}\right). \tag{5.42}$$

Similarly, from the above expressions (5.40) and (5.41) we get again a simple expression for the difference

$$E_s^{(2)} - E_g^{(2)} = -\frac{\tilde{\lambda}^3}{8J^2}\left[13\sum_i^M n_i^4 \sum_j^M n_j^2 - 12\left(\sum_i^M n_i^3\right)^2 + 3\left(\sum_i^M n_i^2\right)^3\right] + O\left(\frac{\tilde{\lambda}^4}{J^2}\right). \tag{5.43}$$

An explanation for the "3-loop" mismatches like (5.42) was suggested in [4]. It was observed there that quantum superstring corrections to energies of *"fast"* semiclassical strings contain non-analytic $\sqrt{\lambda}$ terms[9] (see also [26]). These should effectively promote the coefficient of the quantum $\tilde{\lambda}^3$ corrections into an interpolating function of λ which should have two different limiting values at small (perturbative gauge theory) and large (perturbative string theory) values of λ. This also suggests [4] the presence of such interpolating functions in the S-matrix part of the string Berhe ansatz of [24]. A similar explanation should apply also to $1/J^2$ corrections in (5.43).

6. Non-analytic corrections

In this section we shall discuss how to obtain non-analytic terms of the type found in [4] from the quantum LL hamiltonian. Here we will see that non-analytic terms are present, but it will also be evident that other modes of the superstring are likely to contribute to them.

[8] We do not, however, know how to connect (2.5) directly to the dilatation operators in [33], see [27] for a related discussion.

[9] For "non-fast" semiclassical strings this, of course, is not surprising and was known before [3].

The results of [4] suggest that the $1/J^2$ coefficient function $h_2(\tilde{\lambda})$ in (1.1) should contain terms with half-integer powers of $\tilde{\lambda}$ starting with $\tilde{\lambda}^{5/2}$. The $\tilde{\lambda}^{5/2}/J^2$ contribution should correspond to the first subleading term in the interpolating function $f(\lambda)$ in the quantum string-theory result for the $1/J$ correction to the BMN energies. Indeed, one can generalize (4.7) or (5.42) to

$$E_s^{(1)} = E_g^{(1)} - \frac{\tilde{\lambda}^3}{16J} f(\lambda) \sum_{k,j=1}^{M} n_j^2 n_k^2 (n_k - n_j)^2 + O\Big(\frac{\tilde{\lambda}^4}{J}\Big) , \tag{6.1}$$

$$f(\lambda)_{\lambda \gg 1} = 1 + \frac{a_1}{\sqrt{\lambda}} + \frac{a_2}{(\sqrt{\lambda})^2} + \cdots , \qquad f(\lambda)_{\lambda \to 0} \to 0 . \tag{6.2}$$

Then the presence of the interpolating function $f(\lambda)$ can explain the "3-loop disagreement" found in [5]. Written in terms of $\tilde{\lambda} = \lambda/J^2$ the coefficient in (6.1) is

$$\frac{\lambda^3}{J^7} f(\lambda) = \frac{\tilde{\lambda}^3}{J}\Big(1 + \frac{a_1}{J\sqrt{\tilde{\lambda}}} + \frac{a_2}{J^2(\sqrt{\tilde{\lambda}})^2} + \cdots\Big) = \frac{\tilde{\lambda}^3}{J} + a_1 \frac{\tilde{\lambda}^{5/2}}{J^2} + a_2 \frac{\tilde{\lambda}^2}{J^3} + \cdots . \tag{6.3}$$

We should thus expect to find the non-analytic $\tilde{\lambda}^{5/2}$ term in the string expression for the M-impurity BMN energy at order $1/J^2$, and it should have a simple coefficient proportional to $\sum_{k,j=1}^{M} n_j^2 n_k^2 (n_k - n_j)^2$.[10]

The presence of the functions like $f(\lambda)$ in (6.1) can be related [4] to the presence of the interpolating functions $c_r(\lambda)$ in the phase part of the string Bethe ansatz (B.15): indeed, $c_0(\lambda)$ can then be directly identified with $f(\lambda)$. Assuming universality of the Bethe ansatz, the result of [4] about the 1-loop string correction to the circular string state energy then implies that the first subleading coefficient in f should be $a_1 = -16/3$.

Our aim below will be to see if such non-analytic terms can be captured in the quantum LL approach.[11] That may seem unlikely a priori since in the LL action we certainly miss some string contributions and so are not guaranteed to get the non-analytic terms right (see appendix C); also, the issue may be complicated by the presence of the UV divergences in the LL approach (the full string result is of course finite). More importantly, the string LL action is obtained by taking a large $\mathcal{J} = J/\sqrt{\lambda}$ or, equivalently, small $\tilde{\lambda}$, limit of string theory, and its explicit all-order in $\tilde{\lambda}$ form that generalizes (2.2) is not known at present. However, in the case of the near-BMN expansion it may be sufficient just to use the exact form of the quadratic terms in $\vec{n}$ already included in (2.2) which are known to correctly reproduce the leading BMN spectrum to all orders in $\tilde{\lambda}$. One may then expect that using this exact "kinetic" term while treating other non-linear terms in (2.2) perturbatively may be sufficient to reproduce the non-analytic terms in the near-BMN spectrum.

As discussed in [4], the non-analytic in $\tilde{\lambda}$ corrections in the semiclassical expansion are quantum (as opposed to "finite-size", cf. also [34]) string corrections and they should

[10] To test the presence of other "non-analytic" terms predicted by (6.3), e.g., $a_2 \frac{\tilde{\lambda}^2}{J^3}$, one would need to compute the $1/J^3$ corrections to the BMN energies.

[11] Here we assume of course that the starting point is the LL action with the "string" coefficients in (2.3): the gauge-theory LL action viewed as an effective action corresponding to gauge-theory spin chain should only be treated perturbatively in λ.

come from the large virtual momenta or UV region, i.e. they should be present not only on an $R \times S^1$ world-sheet but also on R^2. To find non-analytic terms in the quantum string expression one may thus replace mode sums by momentum integrals, do all virtual momentum integrations and only then consider the expansion in small $\tilde\lambda$. Expanding first in $\tilde\lambda$ produces (after an appropriate regularization) only analytic corrections.

Replacing the sum over the virtual quantum number q by an integral in the exchange contribution $\frac{(\mathrm{E}_4^{(1)})^2}{W_1}$ in (5.1), (5.6),[12] we get

$$Y = \sum_{i\neq j}^{M} \int_{-\infty}^{\infty} dq \frac{(\mathrm{E}_4^{(1)})^2}{4W_1} \equiv \frac{1}{4J^2} \sum_{i\neq j}^{M} \Big[y(\infty) - y(-\infty) \Big] . \tag{6.4}$$

Here we should be interested in the region of large $q \sim 1/\sqrt{\tilde\lambda}$. To isolate this region we may first set $q = x/\sqrt{\tilde\lambda}$ and then expand the integrand at small $\sqrt{\tilde\lambda}$ for fixed x. Performing the indefinite integral over x we find a series of terms

$$y(x) = B_{-1}\tilde\lambda^{-1/2} + B_0 + B_1\tilde\lambda^{1/2} + B_2\tilde\lambda + B_3\tilde\lambda^{3/2} + B_4\tilde\lambda^2 + B_5\tilde\lambda^{5/2} + B_6\tilde\lambda^3 + B_7\tilde\lambda^{7/2} + \cdots , \tag{6.5}$$

where the coefficients $B_k(x)$ diverge in the limit of large x. These singularities represent the UV divergences coming from the bosonic fields in the SU(2) LL sector. They should be cancelled in the full superstring result against the contributions of other world-sheet fields.

Let us recall that the leading $1/J$ correction to BMN energies computed from the LL action was also divergent and needed to be defined using a particular (normal ordering and zeta-function) regularization in order to match the exact string/gauge theory results [17]. The complication we encounter here is that the above integral contains logarithmic divergencies[13] and so it is unclear a priori how to define its finite part. Thus, starting with the quantum LL action we can indeed confirm the presence of non-analytic contributions, but we are unable to compute their coefficients in an unambiguous way.[14]

Explicitly, using (5.2), (5.3), computing the integral over x, expanding in large x and symmetrising in i, j we find

$$\begin{aligned}
B_{2k} &= O\Big(\frac{1}{x}\Big), \qquad k = 0, 1, 2, 3, \ldots , \\
B_{-1} &= -\frac{1}{2} - \ln(2x) + O\Big(\frac{1}{x^2}\Big), \qquad B_1 = -\frac{1}{2}(n_j^2 - 8n_jn_i + n_i^2) + O\Big(\frac{1}{x^2}\Big), \\
B_3 &= -n_j^2n_i^2(8.04 - 6.87\ln x) + O\Big(\frac{1}{x}\Big),
\end{aligned} \tag{6.6}$$

[12] All other contributions to the $1/J^2$ correction do not contain infinite sums and so cannot produce non-analytic terms.

[13] We did not encounter logarithmic divergences in the computation in section 5.3 where we first expanded in $\tilde\lambda$ and so it was possible to regularize away all power divergences using the ζ-function prescription.

[14] Similar logarithmic divergencies would appear if one would repeat the computation of non-analytic terms in the 1-loop correction to the circular string solution in [4] by keeping in the sum over characteristic frequencies only the contributions from the 2 bosonic modes in the SU(2) sector: logarithmic divergencies are cancelled only after one includes the contributions of all other modes.

$$B_5 = n_j^2 n_i^2\Big[6.31n_j^2 + 2.66n_j n_i + 6.31n_i^2 + (0.16n_j^2 - 0.04n_j n_i + \\ + 0.16n_i^2)\ln x\Big] + O\left(\frac{1}{x}\right), \tag{6.7}$$

$$B_7 = n_j^2 n_i^2\Big[2.28n_j^4 - 19n_j^3 n_i + 10n_j^2 n_i^2 - 19n_j n_k^3 + 2.28n_i^4 - \tag{6.8} \\ - (1.44n_j^4 + 5.39n_j^3 n_i + 2.67n_j^2 n_i^2 + 5.39n_j n_i^3 + 1.44n_i^4)\ln x\Big] + O\left(\frac{1}{x}\right).$$

Here we ignored all power divergencies but kept the logarithmic ones and did not write explicitly subleading $1/x^k$ terms that do not contribute to (6.4).

We conclude that only the terms with *odd* powers of $\sqrt{\tilde{\lambda}}$ receive non-vanishing contributions from the large q region. This is in agreement with the expectation [4] that contributions from this UV region are responsible for the non-analytic terms.

We expect that all logarithmic divergences will get cancellled in the full string computation and also that the finite parts of the coefficients B_{2k} that accompany them will get modified, so that, in particular the first three coefficients B_{-1}, B_1, B_3 will become zero. If that happens, then indeed the leading non-analytic term appearing in the string BMN energy will be proportional to $\frac{\tilde{\lambda}^{5/2}}{J^2}\sum_{i,j} B_5(n_i, n_j)$. These expectations are supported by the results for the 1-loop corrections to the energy of the circular string solution prsented in appendix C.

7. Concluding remarks

The computations from the string LL action are relatively simple, and certainly simpler than the full superstring computations including contributions from all world-sheet fields. However, *a priori*, one cannot guarantee that starting from this truncated string action one will obtain the true superstring result for the $1/J^2$ corrections. A sceptic may say that both the string LL action and the quantum string Bethe ansatz have their origin in the part of string action describing strings already restricted to $R \times S^3$ and, in particular, with no fermion fields present. The hope, however, is that the effect of fermions and other bosonic modes turns out to be relatively benign to the order considered and can be captured by a proper choice of the UV regularization. This hope was indeed realized in several leading-order $1/J$ computations described in [16, 17]. It would still be an important check to carry out the full superstring computation of the $1/J^2$ correction and directly confirm our result (5.41).

The results presented here are also indicative of an underlying integrability. The string LL action is derived from an $R\times S^3$ σ-model that was known to be classically integrable and which effectively has all higher derivative terms included (they appear once one performs the fast-string expansion and solves for the time derivatives of $\vec{n}$, see [19, 20]). Here it seems that we can go a step beyond this and find that the string LL action is consistent with integrability at the first few quantum levels. On the other hand, for the gauge theory we are not starting with a classically integrable LL action with all higher derivatives included, but instead with a proposed two-body S-matrix with an assumed integrability. Although we only have results to $\tilde{\lambda}^3$ order, it is nontrivial that one can construct a lagrangian from this S-matrix that is local in its $\tilde{\lambda}$ series expansion and which is consistent with all predictions from the Bethe ansatz.

It is also interesting that there seem to be at least two distinct integrable LL actions. This is in line with the observation in [33] that there is a family of integrable dilatation operators at λ^3 order. In fact, it might be possible to relate the two actions through some nontrivial mapping. This could perhaps be accomplished using the ideas in [15].

There are several obvious generalizations. It would be interesting to go beyond the $\tilde{\lambda}^3$ order in the LL approach, especially since in (B.13) we already have the all-order Bethe ansatz result. It remains to be seen if one could effectively find the all-order form of the LL action. Another interesting direction would be to go beyond the SU(2) sector and see if the agreement between an effective LL action and the Bethe ansatz can be maintained.

Finally, it would be important to get a better handle on the non-analytic terms, since it is these terms that are ultimately responsible for the apparent disagreements between the perturbative gauge and the perturbative string expressions.

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A. Fixing coefficients in 3-loop gauge theory LL action: circular string example

Here we shall describe how to fix the coefficients (2.4) in the LL action on the gauge theory side. The coefficients **a** and **b** can be found, e.g., by comparing the $1/J$ correction to the BMN energy of a generic M-impurity state with the Bethe ansatz result (see section 4). Here we will also fix **c** using the simplest circular string solution as a test background and then check the consistency of (2.4) with more general circular solutions.

The equal-spin $J_1 = J_2$ circular string in the SU(2) sector [7, 35] corresponds to the simplest non-trivial static solution of the LL action (to all orders in $\tilde{\lambda}$):

$$\vec{n} = (\cos 2m\sigma,\ \sin 2m\sigma,\ 0)\,. \qquad \text{(A.1)}$$

Here m is the integer winding number and $n_i'' = -4m^2 n_i$, so that $n_i'' n_i' = 0$. Thus its energy found from (2.2) does not depend on the value of coefficient **b**. The classical energy of this solution obtained from the "gauge" LL lagrangian (2.2) is (see below)

$$E = J\left[1 + \frac{\tilde{\lambda}}{2}m^2 - \frac{\tilde{\lambda}^2}{8}m^4 + \tilde{\lambda}^3 m^6(1+\mathbf{a}+\mathbf{c}) + O(\tilde{\lambda}^4)\right]. \qquad \text{(A.2)}$$

On the other hand, it was found in [6] from the Bethe ansatz that the three-loop correction to the anomalous dimension of the corresponding spin chain state vanishes. This implies

$$\mathbf{c}_g = -1 - \mathbf{a}_g\,, \qquad \text{i.e.} \qquad \mathbf{c}_g = \frac{3}{4}\,, \qquad \text{(A.3)}$$

where we used that $\mathbf{a}_g = -7/4$ as follows from the matching of the $1/J$ correction to the BMN energy described in section 4.

As a consistency check on the values of the **b** and **c** coefficients in (2.4) let us consider the more general circular solution in the SU(2) sector with $J_1 \neq J_2$ [35]. Its energy to order $\tilde{\lambda}^3$ can be computed from the string theory [35], as well as from the gauge theory Bethe ansatz [23]. Starting with the LL lagrangian (2.2) and plugging the leading order solution into the $\tilde{\lambda}^3$ term in the LL hamiltonian one obtains the correction to the LL energy that can be matched onto either the string or gauge Bethe ansatz result, thus checking the coefficients in (2.3), (2.4).

Let us first recall the details of the rational circular string solution in the SU(2) sector [35]

$$\mathrm{X}_r = a_r e^{i(w_r\tau + m_r\sigma)}\,, \qquad r = 1, 2 \tag{A.4}$$

where $\mathrm{X}_r^2 = 1$ are S^3 coordinates and

$$a_1^2 + a_2^2 = 1\,, \qquad w_r = \sqrt{m_r^2 + \nu^2}\,, \qquad J_r = \sqrt{\lambda}\mathcal{J}_r = \sqrt{\lambda} a_r^2 w_r\,. \tag{A.5}$$

ν is a parameter to be determined from the conformal gauge constraints

$$\mathcal{E}^2 = 2(w_1\mathcal{J}_1 + w_2\mathcal{J}_2) - \nu^2\,, \qquad m_1\mathcal{J}_1 + m_2\mathcal{J}_2 = 0\,, \tag{A.6}$$

where the energy is $E = \sqrt{\lambda}\mathcal{E}$. Introducing the notation

$$m \equiv m_1\,, \qquad n \equiv m_1 - m_2\,, \qquad \mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2\,,$$

one can solve one of the constraints for ν at large $\mathcal{J}$ or small $\tilde{\lambda} = \frac{1}{\mathcal{J}^2}$ to obtain

$$\nu^2 = \mathcal{J}^2 + m(m-n) - \frac{3m(m-n)(2m-n)^2}{4\mathcal{J}^2} + \frac{5m(m-n)(2m-n)^4}{8\mathcal{J}^4} + O\left(\frac{1}{\mathcal{J}^6}\right). \tag{A.7}$$

Then the string energy to $\tilde{\lambda}^3$ order is found to be

$$\begin{aligned} E_s = J\Big[&1 + \frac{1}{2}\tilde{\lambda} m(n-m) - \frac{1}{8}\tilde{\lambda}^2 m(n-m)(n^2 - 3mn + 3m^2) + \\ &+ \frac{1}{16}\tilde{\lambda}^3 m(n-m)(n^4 - 7mn^3 + 20m^2n^2 - 26m^3n + 13m^4) + O(\tilde{\lambda}^4)\Big]. \end{aligned} \tag{A.8}$$

One can also compute the energy of the corresponding state (rational one-cut solution) on the gauge theory side by using the Bethe ansatz as in [23][15]

$$\begin{aligned} E_g = J\Big[&1 + \frac{1}{2}\tilde{\lambda} m(n-m) - \frac{1}{8}\tilde{\lambda}^2 m(n-m)(n^2 - 3mn + 3m^2) + \\ &+ \frac{1}{16}\tilde{\lambda}^3 m(n-m)(n-2m)^2(n^2 - 3mn + 3m^2) + O(\tilde{\lambda}^4)\Big]. \end{aligned} \tag{A.9}$$

[15] For this one is to plug eq. (A.9) in [23] into eq. (5.29) there.

The difference between the two energies to three loops has a simple form (cf. (4.7), see also [4])

$$E_s - E_g = J\left[\frac{\tilde{\lambda}^3}{16}m^3(n-m)^3 + O(\tilde{\lambda}^4)\right]. \tag{A.10}$$

Starting now with the LL lagrangian (2.2) let us find the energy for the corresponding solution with $J_1 \neq J_2$ which is given to leading order by $\vec{n} = (n_1, n_2, n_3)$ where [19]

$$n_1 = 2\sqrt{\frac{m}{n}\left(1-\frac{m}{n}\right)}\cos n\sigma + O(\tilde{\lambda}), \qquad n_2 = 2\sqrt{\frac{m}{n}\left(1-\frac{m}{n}\right)}\sin n\sigma + O(\tilde{\lambda}), \tag{A.11}$$

$$n_3 = 1 - \frac{2m}{n} + O(\tilde{\lambda}). \tag{A.12}$$

This solution can also be found by expanding the full string solution at large $\mathcal{J}$.[16] Plugging this solution into the hamiltonian in (2.2) we find for its LL energy

$$\begin{aligned} E_{LL} = J\Big[&1 + \frac{\tilde{\lambda}}{2}m(n-m) - \frac{\tilde{\lambda}^2}{8}m(n-m)(3m^2+n^2-3mn) + \\ &+ \frac{\tilde{\lambda}^3}{16}m(n-m)\left[n^4 + 4\mathbf{a}\,n^2 m(n-m) + 16\mathbf{c}\,m^2(n-m)^2\right] + O(\tilde{\lambda}^4)\Big]. \end{aligned} \tag{A.13}$$

One can see that for the string values $\mathbf{a}_s = -7/4$, $\mathbf{c}_s = 13/16$ (A.13) reproduces the string energy (A.8), while for the gauge values $\mathbf{a}_g = -7/4$, $\mathbf{c}_g = 3/4$ (A.13) reproduces the gauge theory result (A.9).

B. $1/J^2$ corrections to energy of M-impurity states from the Bethe ansatz

In this appendix we compute the energies of an M-impurity state up to and including $1/J^2$ corrections for the gauge [14] and string [24] all-loop spin chain.

B.1 Gauge theory

Starting with the BDS Bethe equations [14]

$$e^{ip_i J} = \prod_{j\neq i}^{M} \frac{u_i - u_j + i}{u_i - u_j - i}, \tag{B.1}$$

where

$$u = \frac{1}{2}\cot\frac{p}{2}\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}, \tag{B.2}$$

we can write (B.1) up to order $1/J^2$ accuracy as

$$e^{ip_i J} = \exp\left[\sum_{j\neq i}^{M}\frac{2i}{u_i - u_j}\right]. \tag{B.3}$$

[16] The unit vector $\vec{n}$ can be written as $\vec{n} = (\sin 2\psi\cos 2\varphi, \sin 2\psi \sin 2\varphi, \cos 2\psi)$. In terms of global angular coordinates of S^5 with the metric $ds^2 = dt^2 + d\gamma^2 + \cos^2\gamma\, d\varphi_3^2 + \sin^2\gamma\,(d\psi^2 + \cos^2\psi\, d\varphi_1^2 + \sin^2\psi\, d\varphi_2^2)$ we have $\varphi = \frac{\varphi_1 - \varphi_2}{2}$. Note also that the cartesian coordinates are $\mathrm{X}_1 = \cos\psi\, e^{i\varphi_1}$, $\mathrm{X}_2 = \sin\psi\, e^{i\varphi_2}$.

If we now set

$$p_i = \frac{2\pi n_i}{J} + \Delta_i \,, \tag{B.4}$$

then u_i can be approximated as

$$u_i \approx w_i \left(\frac{J}{2\pi n_i}\right) - \frac{\Delta_i}{w_i}\left(\frac{J}{2\pi n_i}\right)^2, \qquad w_i \equiv \sqrt{1+\tilde{\lambda} n_i^2}\,, \tag{B.5}$$

and so Δ_i satisfies to $1/J^2$ accuracy

$$\Delta_i = \frac{4\pi}{J^2}\sum_{j\neq i}\left[\frac{n_i n_j}{\theta_{ji}} + \frac{J}{2\pi}\frac{n_j^2\Delta_i w_j - n_i^2 \Delta_j w_i}{w_i w_j (\theta_{ji})^2}\right], \qquad \theta_{ji} \equiv n_j w_i - n_i w_j \,. \tag{B.6}$$

We will assume that all n_i are different. Up to the desired order, we can write $\Delta_i = \Delta_i^{(1)} + \Delta_i^{(2)}$, where

$$\Delta_i^{(1)} = \frac{4\pi}{J^2}\sum_{j\neq i}^{M}\frac{n_i n_j}{\theta_{ji}}\,, \tag{B.7}$$

and

$$\begin{aligned}\Delta_i^{(2)} = \frac{8\pi}{J^3}\Bigg[&\sum_{j\neq i}^{M}\frac{n_i n_j (n_i^2 w_i + n_j^2 w_j)}{w_i w_j (\theta_{ji})^3} + \\ &+ \sum_{\substack{k\neq j\neq i\\ k\neq i}}^{M}\frac{n_i n_j n_k (n_k n_j w_j^2 - n_k n_i w_i^2 - n_j^2 w_k w_j + n_i^2 w_k w_j)}{w_i w_j \theta_{ki}\theta_{kj}(\theta_{ji})^2}\Bigg].\end{aligned} \tag{B.8}$$

Now the energy is given by

$$\begin{aligned}E &= \sum_i^M\left(\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_i}{2}} - 1\right) \\ &\approx \sum_i^M\left(\sqrt{1+\frac{\lambda}{4\pi^2}p_i^2} - 1 - \frac{\lambda p_i^4}{96\pi^2\sqrt{1+\frac{\lambda}{4\pi^2}p_i^2}}\right).\end{aligned} \tag{B.9}$$

Hence as an expansion in $1/J$ we find

$$E^{(0)} = \sum_i^M (w_i - 1)\,, \tag{B.10}$$

$$E^{(1)} = \frac{J\tilde{\lambda}}{2\pi}\sum_i^M \frac{\Delta_i^{(1)} n_i}{w_i} = -\frac{\tilde{\lambda}}{J}\sum_{j\neq i}^M \frac{n_i n_j}{w_i w_j}\,, \tag{B.11}$$

and

$$E^{(2)} = -\frac{(2\pi)^2\tilde{\lambda}}{24J^2}\sum_i^M \frac{n_i^4}{w_i} + \frac{J\tilde{\lambda}}{2\pi}\sum_i^M\frac{\Delta_i^{(2)} n_i}{w_i} + \frac{J^2\tilde{\lambda}}{8\pi^2}\sum_i^M\frac{(\Delta_i^{(1)})^2}{w_i^3}$$

$$
= -\frac{\pi^2\tilde{\lambda}}{6J^2}\sum_i^M \frac{n_i^4}{w_i} + \frac{4\tilde{\lambda}}{J^2}\left[\sum_{j\neq i}^M \frac{n_i n_j (n_i^2 w_i + n_j^2 w_j)}{w_i w_j (\theta_{ji})^2} + \right.
$$

$$
\left. + \sum_{k\neq j\neq i\neq k}^M \frac{n_i n_j n_k (n_k n_j w_j^2 - n_k n_i w_i^2 - n_j^2 w_k w_j + n_i^2 w_k w_j)}{w_i w_j \theta_{ki}\theta_{kj}(\theta_{ji})^2}\right] +
$$

$$
+\frac{2\tilde{\lambda}}{J^2}\sum_i^M \sum_{\substack{j\neq i\\ k\neq i}}^M \frac{n_i^2 n_j n_k}{w_i^3 \theta_{ji}\theta_{ki}}\,. \tag{B.12}
$$

Symmetrizing the sums, and splitting the last term into a piece where $k = j$ and another piece where $k \neq j$, we find

$$
\begin{aligned}
E^{(2)} &= -\frac{\pi^2\tilde{\lambda}}{6J^2}\sum_i^M \frac{n_i^4}{w_i} - \frac{\tilde{\lambda}}{J^2}\sum_{j\neq i}^M \frac{n_i n_j (2n_i^2 w_i^2 w_j - n_i n_j (w_i^3 + w_j^3) + 2n_j^2 w_j^2 w_i)}{w_i^3 w_j^3 (\theta_{ji})^2} + \\
&\quad + \frac{2\tilde{\lambda}}{3J^2}\sum_{k\neq j\neq i\neq k}^M \frac{n_i n_j n_k}{(w_i w_j w_k)^2 \theta_{ki}\theta_{ij}\theta_{jk}} \times \\
&\quad \times \left\{ +(n_i^2 - n_j^2) w_i w_j + (n_j^2 - n_k^2) w_j w_k + (n_k^2 - n_i^2) w_k w_i + \right. \\
&\quad \left. + \frac{n_i n_j (w_i^3 - w_j^3) w_k^4 + n_j n_k (w_j^3 - w_k^3) w_i^4 + n_k n_i (w_k^3 - w_i^3) w_i^4}{w_i w_j w_k} \right\}.
\end{aligned} \tag{B.13}
$$

Note that the last two terms in (B.13) are zero in the one-loop limit, where all $w_i = 1$.

Expanding the $1/J^2$ term in (B.13) to order $\tilde{\lambda}^3$, we find using the momentum constraint $\sum_i^M n_i = 0$

$$
\begin{aligned}
E_g^{(2)} &= \frac{\tilde{\lambda}}{J^2}\left[-\frac{\pi^2}{6}\sum_i^M n_i^4 + 2\sum_i^M n_i^2 - \sum_{j\neq i}^M \frac{2n_i^2 n_j^2}{(n_i - n_j)^2}\right] + \\
&\quad + \frac{\tilde{\lambda}^2}{J^2}\left[\frac{\pi^2}{12}\sum_i^M n_i^6 - \frac{13}{2}\sum_i^M n_i^4 + \frac{3}{2}\left(\sum_i^M n_i^2\right)^2 + \sum_{i\neq j}^M \frac{n_i^3 n_j^3}{(n_i - n_j)^2}\right] + \\
&\quad + \frac{\tilde{\lambda}^3}{J^2}\left[-\frac{\pi^2}{16}\sum_i^M n_i^8 + \frac{49}{4}\sum_i^M n_i^6 - \frac{9}{4}\left(\sum_i^M n_i^4\right)\left(\sum_i^M n_i^2\right) - \frac{9}{2}\left(\sum_i^M n_i^3\right)^2 - \right. \\
&\quad \left. - \frac{1}{4}\left(\sum_i^M n_i^2\right)^3 - \frac{3}{4}\sum_{i\neq j}^M \frac{n_i^4 n_j^4}{(n_i - n_j)^2}\right] + \mathrm{O}\left(\frac{\tilde{\lambda}^4}{J^2}\right).
\end{aligned} \tag{B.14}
$$

B.2 String theory

Let us now consider the proposed string Bethe ansatz in [24]. In this case the Bethe equations in (B.1) are modified to

$$
e^{ip_i J} = \prod_{j\neq i}^M \frac{u_i - u_j + i}{u_i - u_j - i}\exp\left(-2i\sum_{r=0}^\infty \left(\frac{\lambda}{16\pi^2}\right)^{r+2}\left[q_{r+3}(p_i) q_{r+2}(p_j) - q_{r+3}(p_j) q_{r+2}(p_i)\right]\right), \tag{B.15}
$$

where $q_r(p)$ is one of the higher charges of an impurity with momentum p and is given by

$$q_r(p) = \frac{2\sin(\frac{1}{2}(r-1)p)}{r-1}\left(\frac{\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{1}{2}p}-1}{\frac{\lambda}{4\pi^2}\sin\frac{1}{2}p}\right)^{r-1}. \tag{B.16}$$

For the accuracy desired here, we may approximate this as

$$q_r(p) \approx \left(\frac{8\pi^2}{\lambda}\right)^{r-1}\frac{1}{p^{r-2}}\tilde{w}^{r-1}, \qquad \tilde{w} \equiv \sqrt{1+\frac{\lambda}{4\pi^2}p^2}-1\,. \tag{B.17}$$

With this approximation we can write the extra term in (B.15) as

$$-2i\sum_{r=0}^{\infty}\Big[q_{r+3}(p_i)q_{r+2}(p_j)-q_{r+3}(p_j)q_{r+2}(p_i)\Big] = -i\frac{\tilde{w}_i\tilde{w}_j(p_j\tilde{w}_i-p_i\tilde{w}_j)}{p_ip_j\frac{\lambda}{4\pi^2}-\tilde{w}_i\tilde{w}_j}, \tag{B.18}$$

which after much manipulation can be simplified to

$$i(p_i-p_j)-i\frac{p_i^2\bar{w}_j-p_j^2\bar{w}_i}{p_i+p_j}, \qquad \bar{w}_i \equiv \sqrt{1+\frac{\lambda}{4\pi^2}p_i^2}\,. \tag{B.19}$$

We thus find the equation

$$Jp_i - 2\pi n_i = \sum_{j\neq i}^{M}\left(\frac{2p_ip_j}{p_j\bar{w}_i-p_i\bar{w}_j}+p_i-p_j-\frac{p_i^2\bar{w}_j-p_j^2\bar{w}_i}{p_i+p_j}\right). \tag{B.20}$$

Since the extra term found in the string computation does not have a pole at $p_i = p_j$, the residue of the double pole in (B.13) will not be affected. If one wants to keep all orders in $\tilde{\lambda}$, then it is convenient to replace the sum over $p_i - p_j$ in (B.20) with Mp_i, which follows from the zero momentum condition. Then we find that the more natural expansion parameter is $1/J'$ where $J' = J - M$. However, to compute terms in the Taylor series in $\tilde{\lambda}$, which is what we will do here, it is easier to leave (B.20) as it is. If we now substitute (B.4) into (B.20) we find the equation

$$\begin{aligned}\Delta_i = \frac{4\pi}{J^2}\sum_{j\neq i}\Bigg[&\frac{n_in_j}{\theta_{ji}}+\frac{J}{2\pi}\frac{n_j^2\Delta_iw_j-n_i^2\Delta_jw_i}{w_iw_j(\theta_{ji})^2}+\\ &+\frac{1}{2}\Bigg\{n_i-n_j-\frac{n_i^2w_j-n_j^2w_i}{n_i+n_j}+\\ &\qquad+\frac{J}{2\pi}\Bigg(\Delta_i-\Delta_j-\Delta_iw_j\frac{n_j^2(1-n_in_j\tilde{\lambda})+n_i(n_i+2n_j)w_iw_j}{w_iw_j(n+m)^2}+\\ &\qquad\qquad+\Delta_jw_i\frac{n_i^2(1-n_in_j\tilde{\lambda})+n_j(n_j+2n_i)w_iw_j}{w_iw_j(n+m)^2}\Bigg)\Bigg\}\Bigg].\end{aligned} \tag{B.21}$$

where the term in the curly brackets is the extra piece coming from the string QBA. Again writing $\Delta_i = \Delta_i^{(1)} + \Delta_i^{(2)}$, we find that

$$\Delta_i^{(1)} = \frac{4\pi}{J^2}\sum_{j\neq i}^{M}\left(\frac{n_in_j}{\theta_{ji}}+\frac{n_i-n_j}{2}-\frac{n_i^2w_j-n_j^2w_i}{2(n_i+n_j)}\right). \tag{B.22}$$

The expression for $\Delta_i^{(2)}$ is quite lengthy so we will not present the result here. In any case, the expressions for $\Delta^{(1)}$ and $\Delta^{(2)}$ can then be plugged directly into the first line of (B.12). The final result for the $1/J^2$ correction to the energy, up to order $\tilde\lambda^3$ is

$$\begin{aligned}
E_s^{(2)} = & \frac{\tilde\lambda}{J^2}\left[-\frac{\pi^2}{6}\sum_i^M n_i^4 + 2\sum_i^M n_i^2 - \sum_{j\neq i}^M \frac{2n_i^2 n_j^2}{(n_i-n_j)^2}\right] + \\
& +\frac{\tilde\lambda^2}{J^2}\left[\frac{\pi^2}{12}\sum_i^M n_i^6 - \frac{13}{2}\sum_i^M n_i^4 + \frac{3}{2}\left(\sum_i^M n_i^2\right)^2 + \sum_{i\neq j}^M \frac{n_i^3 n_j^3}{(n_i-n_j)^2}\right] + \\
& +\frac{\tilde\lambda^3}{J^2}\Bigg[-\frac{\pi^2}{16}\sum_i^M n_i^8 + \frac{49}{4}\sum_i^M n_i^6 - \frac{31}{8}\left(\sum_i^M n_i^4\right)\left(\sum_i^M n_i^2\right) - 3\left(\sum_i^M n_i^3\right)^2 + \\
& \qquad + \frac{1}{8}\left(\sum_i^M n_i^2\right)^3 - \frac{3}{4}\sum_{i\neq j}^M \frac{n_i^4 n_j^4}{(n_i-n_j)^2}\Bigg] + \mathrm{O}\left(\frac{\tilde\lambda^4}{J^2}\right).
\end{aligned} \tag{B.23}$$

Note that the order $\tilde\lambda$ and $\tilde\lambda^2$ terms here agree with the gauge theory result in (B.14), as expected. At order $\tilde\lambda^3$ there are differences, even though there is agreement for the pole term and the $\sum n_i^6$ and $\sum n_i^8$ terms.

C. Non-analytic terms in one-loop correction to energy of circular string solution

Here we shall provide some details about the structure of non-analytic terms in one-loop correction to the energy of $J_1 = J_2$ circular solution of [7, 35] considered in [4]. Our motivation is to further illustrate the observation of section 6 that restriction to the modes which are present in the LL sector while ignoring all other superstring modes does reveal the existence of the non-analytic in $\tilde\lambda$ terms but does not allow to determine them in an unambiguous way.

The one-loop correction to the energy has the form [10, 16][17]

$$E_1 = \frac{1}{2\kappa}\sum_{n=-\infty}^{\infty}\Big[\mathcal{S}_{LL}(n,\kappa,m) + \mathcal{S}_{\rm other}(n,\kappa,m)\Big]. \tag{C.1}$$

Here

$$\begin{aligned}
\mathcal{S}_{LL} = & \left[n^2 + 2\kappa^2 - 2m^2 + 2\sqrt{(\kappa^2-m^2)^2 + n^2\kappa^2}\right]^{1/2} + \\
& + \left[n^2 + 2\kappa^2 - 2m^2 - 2\sqrt{(\kappa^2-m^2)^2 + n^2\kappa^2}\right]^{1/2} \\
= & \sqrt{(n+\sqrt{n^2-4m^2})^2 + 4\kappa^2}
\end{aligned} \tag{C.2}$$

[17] We consider the solution in the form of [35] when fermionic fluctuations are periodic (see also [16]). In fact, the choice of periodicity of the fermions does not actually influence the form of the non-analytic terms.

is the contribution of the two fluctuations from the SU(2) LL sector (bosonic fluctuations in S^3 part of S^5 where the string is rotating, with third — "fast"– coordinate being fixed), while

$$\mathcal{S}_{\rm other} = 4\sqrt{n^2+\kappa^2} + 2\sqrt{n^2+\kappa^2-2m^2} - 8\sqrt{n^2+\kappa^2-m^2} \tag{C.3}$$

is the contribution of 4 AdS_5 fluctuations ($t = \kappa\tau$ is not fluctuating), 2 other modes of S^5 outside of S^3, and of 8 fermionic fluctuation modes. We use the parameters

$$\kappa^2 = \mathcal{J}^2 + m^2\,, \qquad \tilde{\lambda} = \frac{1}{\mathcal{J}^2}\,, \tag{C.4}$$

where m is an integer winding number.

Our aim is to find the terms of *odd* powers in $1/\mathcal{J}$ in the expansion of E_1 at large $\mathcal{J}$, i.e. the terms non-analytic in $\tilde{\lambda}$. As discussed in [4], it is sufficient for this to replace the sum over n in (C.1) by an integral: the terms that correct the integral to the sum happen to be analytic in $\tilde{\lambda}$, i.e. contain even powers of $1/\mathcal{J}$ in their expansion.

Then we get $E_1 = \tilde{E}_1$+analytic terms, where

$$\tilde{E}_1 = \tilde{E}_{LL} + \tilde{E}_{\rm other} = \frac{1}{2\kappa}\int_{-\infty}^{\infty} dn \Big[\mathcal{S}_{LL}(n,\kappa,m) + \mathcal{S}_{\rm other}(n,\kappa,m)\Big]\,. \tag{C.5}$$

Since each term in the integrand is symmetric under $n \to -n$ we can restrict the integral to $(0,\infty)$ and multiply by 2. Introducing a cutoff Λ at large-N we then get

$$\tilde{E}_{LL} = \frac{1}{\kappa}\int_0^{\Lambda} dn\, \mathcal{S}_{LL}(n,\kappa,m)\,, \qquad \tilde{E}_{\rm other} = \frac{1}{\kappa}\int_0^{\Lambda} dn\, \mathcal{S}_{\rm other}(n,\kappa,m)\,. \tag{C.6}$$

The dependence on $\Lambda \to \infty$ will of course cancels in the sum in (C.5) but we would like to study the contribution $\tilde{E}_{LL}$ separately from $\tilde{E}_{\rm other}$.

The integral over n in $\tilde{E}_{LL}$ can be performed explicitly by changing the variable $n = \frac{y^2+4m^2}{2y}$, $y = n + \sqrt{n^2-4m^2}$. To avoid the tachyonic instability coming from few lowest LL frequencies we may compute $\tilde{E}_{LL}$ by first formally setting $m \to im$, computing the integral and rotating back in the final expression. The imaginary part of the integral will contain only analytic terms, while the non-analytic terms will be real: they are not sensitive to this instability having their origin in the "stable" large-N region of the spectrum.

The resulting expression is (the first square bracket is the contribution of the upper limit $n = \Lambda$ in the integral and the second of the $n = 0$ point; we omit all subleading $1/\Lambda$ contributions)

$$\begin{aligned}\tilde{E}_{LL} = \frac{1}{\kappa}\bigg\{ & \Big[\Lambda^2 + \frac{1}{2}\kappa^2 + (\kappa^2-2m^2)[\ln(2\Lambda)+\ln 2]\Big] - \\ & - \Big[- im\sqrt{\kappa^2-m^2} + (\kappa^2-2m^2)\ln[2(\sqrt{\kappa^2-m^2}+im)]\Big]\bigg\}\,.\end{aligned} \tag{C.7}$$

Similarly, we find for the contribution of all other modes

$$\begin{aligned}\tilde{E}_{\rm other} = \frac{1}{\kappa}\bigg\{ & \Big[-\Lambda^2 - (\kappa^2-2m^2)\Big[\ln(2\Lambda) + \frac{1}{2}\Big]\Big] - \\ & - \Big[\kappa^2\ln\kappa^2 + \frac{1}{2}(\kappa^2-2m^2)\ln(\kappa^2-2m^2) - 2(\kappa^2-m^2)\ln(\kappa^2-m^2)\Big]\bigg\}\,.\end{aligned} \tag{C.8}$$

Summing these two contributions together we find that the quadratic and logarithmic divergences indeed cancel. Using (C.4) we may then write the result as the sum of the two combinations that produce, respectively, even and odd terms in the $1/\mathcal{J}$ expansion of E_1

$$\begin{aligned}
\tilde{E}_1 &= \tilde{E}_{\rm even} + \tilde{E}_{\rm odd}\,, && \text{(C.9)}\\
\tilde{E}_{\rm even} &= \frac{1}{\sqrt{\mathcal{J}^2+m^2}}\left[i\mathcal{J}m + \frac{1}{2}(\mathcal{J}^2-m^2)\ln\frac{\mathcal{J}-im}{\mathcal{J}+im}\right], && \text{(C.10)}\\
\tilde{E}_{\rm odd} &= \frac{1}{\sqrt{\mathcal{J}^2+m^2}}\left[m^2 + 2\mathcal{J}^2\ln\frac{\mathcal{J}^2}{\mathcal{J}^2+m^2} - \frac{1}{2}(\mathcal{J}^2-m^2)\ln\frac{\mathcal{J}^2-m^2}{\mathcal{J}^2+m^2}\right], && \text{(C.11)}
\end{aligned}$$

where $\tilde{E}_{odd}$ is the same as eq.(11) in [4], so that

$$\tilde{E}_{\rm odd} = -\frac{m^6}{3\mathcal{J}^5} + \frac{m^8}{3\mathcal{J}^7} - \frac{49m^{10}}{120\mathcal{J}^9} + O(\frac{1}{\mathcal{J}^{11}})\,. \tag{C.12}$$

Now, if instead we concentrate just on the LL sector contribution in (C.7), take its real part, subtract (for simplicity) the power divergence but keep the logarithmic one, set $\Lambda = \mathcal{J}\tilde{\Lambda}$ and expand in large $\mathcal{J}$ for fixed $\tilde{\Lambda}$ we find:

$$\begin{aligned}
\mathrm{Re}(\tilde{E}_{LL}) &= \frac{1}{\sqrt{\mathcal{J}^2+m^2}}\left[\frac{1}{2}(\mathcal{J}^2+m^2) + (\mathcal{J}^2-m^2)\ln(2\Lambda) - \frac{1}{2}(\mathcal{J}^2-m^2)\ln(\mathcal{J}^2+m^2)\right]\\
&= \mathcal{J}\left(\frac{1}{2}+\ln(2\tilde{\Lambda})\right) - \frac{m^2}{4\mathcal{J}}(1+6\ln(2\tilde{\Lambda})) + \frac{m^4}{16\mathcal{J}^3}(15+14\ln(2\tilde{\Lambda}))\\
&\quad - \frac{m^6}{96\mathcal{J}^5}(91+66\ln(2\tilde{\Lambda})) + \frac{5m^8}{256\mathcal{J}^7}(47+30\ln(2\tilde{\Lambda})) + O\left(\frac{1}{\mathcal{J}^9}\right).
\end{aligned} \tag{C.13}$$

Comparing this to (C.12) we conclude that the LL sector produces contributions to the odd terms that contain also order $\mathcal{J}, 1/\mathcal{J}, 1/\mathcal{J}^3$ terms that cancel against similar terms in $\tilde{E}_{\rm other}$ in the total superstring expression. The coefficients of the higher $1/\mathcal{J}^5, 1/\mathcal{J}^7, \ldots$ terms contain cut-off dependent parts that again cancel in (C.12). If one formally ignores the singular $\ln(2\tilde{\Lambda})$ parts, the finite parts of the $1/\mathcal{J}^5, 1/\mathcal{J}^7, \ldots$ coefficients in (C.13) are still different from the ones in (C.12), implying that *all* superstring modes contribute to the finite parts of the coefficients of the odd (non-analitic in $\tilde{\lambda}$) terms. These observations provide support to the interpretation of the results of section 6.

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